Quadrature in Besov spaces on the Euclidean sphere

K. Hesse*
School of Mathematics, University of New South Wales,
Sydney, NSW 2052, Australia

H. N. Mhaskar†
Department of Mathematics, California State University
Los Angeles, California, 90032, U.S.A.

I. H. Sloan‡
School of Mathematics, University of New South Wales,
Sydney, NSW 2052, Australia

Abstract

Let $q \geq 1$ be an integer, $\mathbb{S}^q$ denote the unit sphere embedded in the Euclidean space $\mathbb{R}^{q+1}$, and μ_q be its Lebesgue surface measure. We establish upper and lower bounds for

$$\sup_{f \in \mathcal{B}^\gamma_{p,\rho}} \left| \int_{\mathbb{S}^q} f d\mu_q - \sum_{k=1}^{M} w_k f(\mathbf{x}_k) \right|, \qquad \mathbf{x}_k \in \mathbb{S}^q,\ w_k \in \mathbb{R},\ k = 1, \cdots, M,$$

where $\mathcal{B}^\gamma_{p,\rho}$ is the unit ball of a suitable Besov space on the sphere. The upper bounds are obtained for choices of $\mathbf{x}_k$ and w_k that admit exact quadrature for spherical polynomials of a given degree, and satisfy a certain continuity condition; the lower bounds are obtained for the infimum of the above quantity over all choices of $\mathbf{x}_k$ and w_k. Since the upper and lower bounds agree with respect to order, the complexity of quadrature in Besov spaces on the sphere is thereby established.

*The support of the Australian Research Council is gratefully acknowledged. Part of the work was carried out while the author was a guest of the Center for Constructive Approximation at Vanderbilt University.

†The research of this author was supported, in part, by grant W911NF-04-1-0339 from the U.S. Army Research Office, grant DMS-0204704, and its continuation grant DMS-0605209 from the National Science Foundation.

‡The support of the Australian Research Council is gratefully acknowledged.

1 Introduction

Let $q \geq 1$ be an integer, $\mathbb{S}^q$ be the unit sphere embedded in the Euclidean space $\mathbb{R}^{q+1}$, and μ_q be its Lebesgue surface measure, so that

$$\omega_q := \int_{\mathbb{S}^q} d\mu_q = \frac{2\pi^{(q+1)/2}}{\Gamma((q+1)/2)}. \tag{1.1}$$

For integer $N \geq 0$, let Π_N^q denote the class of all spherical polynomials of degree at most N; i.e., the class of restrictions to $\mathbb{S}^q$ of polynomials in $q+1$ variables with total degree at most N.

Many applications in geophysics and partial differential equations require an approximate evaluation of an integral of the form $\int_{\mathbb{S}^q} f d\mu_q$. Therefore, several mathematicians have recently investigated quadrature formulas for such integrals that are required to be exact for $f \in \Pi_N^q$ and high integer values of N. The examples include the product Gaussian formulas in the book [27, Chapter 2] of Stroud, Driscoll-Healy formulas [6, Theorem 3], and some newer formulas by Brown, Dai and Sheng [2, Theorem 4], Potts, Steidl, and Tasche [22]. Numerically stable interpolatory quadrature formulas based on points that maximize a certain determinant have been studied by Sloan and Womersley [25, Sections 4 and 5]. Jetter, Stöckler, and Ward [12] have established the existence of signed quadrature rules based on “scattered data”; i.e., in the case when no assumptions are made on the location of the nodes. The existence of positive quadrature formulas based on scattered data is proved in [17, Section 4, Theorem 4.1]. Some ideas on the computation of these formulas are also given in [17]. The number of data points for which quadrature formulas, exact for Π_N^q, can be obtained is proportional to the dimension of Π_N^q.

For a quadrature formula of the form $Qf = \sum_{k=1}^M w_k f(\mathbf{x}_k)$, where $w_k \in \mathbb{R}$ and $\mathbf{x}_k \in \mathbb{S}^q$, that is exact for polynomials in Π_N^q, an obvious error bound can be obtained by observing that for any $P \in \Pi_N^q$,

$$\left|\int_{\mathbb{S}^q} f d\mu_q - Qf\right| = \left|\int_{\mathbb{S}^q} (f-P) d\mu_q - Q(f-P)\right| \leq \left(\omega_q + \sum_{k=1}^M |w_k|\right) \max_{\mathbf{x}\in\mathbb{S}^q} |f(\mathbf{x}) - P(\mathbf{x})|.$$

Therefore, if f is a continuous function on $\mathbb{S}^q$, and

$$E_{N,\infty}(f) := \inf\left\{\max_{\mathbf{x}\in\mathbb{S}^q} |f(\mathbf{x}) - P(\mathbf{x})| \ : \ P \in \Pi_N^q\right\},$$

we have

$$\left|\int_{\mathbb{S}^q} f d\mu_q - Qf\right| \leq \left(\omega_q + \sum_{k=1}^M |w_k|\right) E_{N,\infty}(f). \tag{1.2}$$

The rate of convergence of $E_{N,\infty}(f)$ to zero as $N \to \infty$ is traditionally expressed in terms of moduli of smoothness of derivatives of f (see for example Ragozin [23, Theorem 3.4]). A recent result [8, Proposition 4.3 (b)] directly in terms of derivatives is that, if $\gamma > 0$, and f has a continuous fractional derivative $\partial_\gamma f$ of order γ (see (2.7) for a precise definition), then $E_{N,\infty}(f) = \mathcal{O}(N^{-\gamma})$. The same estimate on the quadrature error was proved in the case $q = 2$ for quadrature formulas satisfying a continuity condition, by Hesse and Sloan [10, Theorem 5], under the weaker assumption that $\partial_\gamma f$ is square–integrable on $\mathbb{S}^q$. We note that for all the quadrature formulas mentioned above, this rate is $O(M^{-\gamma/q})$ in terms of the number of nodes M if $M = \mathcal{O}(N^q)$. For $q = 2$, Hesse and Sloan have proved in [11, Theorem 1] that this estimate is the best possible for any quadrature formula using M nodes, without any further assumptions on the nodes, the weights, or on polynomial exactness, by proving a lower bound for the worst–case quadrature error of the same order. These results have been extended to the case of higher values of q in [1, 9] and further to the case of arbitrary L^p spaces and weighted integrals in [16].

The condition in [10, 11] that $\partial_\gamma f$ be square–integrable can be reformulated to state that f is in a certain Besov space (see Proposition 2). In this paper, we study the error of quadrature formulas for arbitrary L^p–Besov spaces on the sphere, $1 \le p \le \infty$. We obtain both upper and lower bounds for this error. The upper bounds are given in terms of the degree of polynomials for which the quadrature formulas are exact. The lower bounds are given in terms of certain geometric quantities associated with the support of the quadrature measures. Since the upper and lower bounds agree with respect to orders of M, they yield a result on the complexity of quadrature in Besov spaces on the sphere.

In Section 2, we develop the notations necessary for stating and proving the results in the paper, as well as review a few known facts. The main results are stated in Section 3, and the proofs are given in Section 4. Essential tools in our proofs are the concept of polynomial frames and their relationship with the Besov spaces, using material that is scattered in various other papers [14, 15, 19, 8, 16]. The Appendix contains a sketch of the proof of Theorem 3, and of the estimate (4.3) used in the proof of Lemma 8.

2 Notations and preliminaries

2.1 General notations

If $1 \le p \le \infty$, and $f : \mathbb{S}^q \to \mathbb{R}$ is Lebesgue measurable, we write

$$\|f\|_p := \begin{cases} \{\int_{\mathbb{S}^q} |f(\mathbf{x})|^p d\mu_q(\mathbf{x})\}^{1/p}, & \text{if } 1 \le p < \infty, \\ \text{ess sup}_{\mathbf{x}\in\mathbb{S}^q} |f(\mathbf{x})|, & \text{if } p = \infty. \end{cases}$$

The space of all Lebesgue measurable functions on $\mathbb{S}^q$ such that $\|f\|_p < \infty$ will be denoted by L^p, with the usual convention that two functions are considered equal as elements of this space if they are equal almost everywhere. The symbol X^p will denote L^p if $1 \le p < \infty$ and the space of all continuous functions on $\mathbb{S}^q$ if $p = \infty$ (equipped with the norm of L^∞). Strictly speaking, the space L^p consists of equivalence classes. If $f \in L^p$ is almost everywhere equal to a continuous function, we will assume that this continuous function is chosen as the representer of its class.

For a fixed integer $\ell \ge 0$, the restriction to $\mathbb{S}^q$ of a homogeneous harmonic polynomial of exact degree ℓ is called a spherical harmonic of degree ℓ. Most of the following information is based on [20], [26, Section IV.2], and [7, Chapter XI], although we use a different notation. The class of all spherical harmonics of degree ℓ will be denoted by $\mathbf{H}^q_\ell$. The spaces $\mathbf{H}_\ell^q$ are mutually orthogonal relative to the inner product of L^2. For any integer $N \ge 0$, we have $\Pi_N^q = \bigoplus_{\ell=0}^N \mathbf{H}_\ell^q$. The dimension of $\mathbf{H}_\ell^q$ is given by

$$d_\ell^q := \dim \mathbf{H}_\ell^q = \begin{cases} \dfrac{2\ell+q-1}{\ell+q-1}\dbinom{\ell+q-1}{\ell}, & \text{if } \ell \ge 1, \\ \qquad\qquad 1, & \text{if } \ell = 0. \end{cases} \tag{2.1}$$

and that of Π_N^q is $\sum_{\ell=0}^N d_\ell^q = d_N^{q+1}$. Furthermore, $L^2 = L^2$–closure$\{\bigoplus_{\ell=0}^\infty \mathbf{H}_\ell^q\}$. Hence, if we choose an orthonormal basis $\{Y_{\ell,k} : k = 1, \cdots, d_\ell^q\}$ for each $\mathbf{H}_\ell^q$, then the set $\{Y_{\ell,k} : \ell = 0, 1, \cdots$ and $k = 1, \cdots, d_\ell^q\}$ is a complete orthonormal basis for L^2. One has the well-known addition formula [20] and [7, Chapter XI, Theorem 4]:

$$\sum_{k=1}^{d_\ell^q} Y_{\ell,k}(\mathbf{x})Y_{\ell,k}(\mathbf{y}) = \frac{d_\ell^q}{\omega_q}\mathcal{P}_\ell(q+1; \mathbf{x}\cdot\mathbf{y}), \qquad \ell = 0, 1, \cdots,$$

where $\mathcal{P}_\ell(q+1; \circ)$ is the degree–ℓ Legendre polynomial in $q+1$-dimensions.

The Legendre polynomials are normalized so that $\mathcal{P}_\ell(q+1; 1) = 1$, and satisfy the orthogonality relations [20, Lemma 10]

$$\int_{-1}^1 \mathcal{P}_\ell(q+1; t)\mathcal{P}_k(q+1; t)(1-t^2)^{\frac{q}{2}-1}dt = \frac{\omega_q}{\omega_{q-1}d_\ell^q}\delta_{\ell,k}.$$

They are related to the ultraspherical (Gegenbauer) polynomials $P_\ell^{(\frac{q-1}{2})}$ (cf. [28, Section 4.7], [20, p. 33]), and the Jacobi polynomials [28, Chapter IV], $P_\ell^{(\alpha,\beta)}$ with $\alpha = \beta = \frac{q}{2} - 1$, via

$$\begin{aligned} P_\ell^{(\frac{q-1}{2})}(t) &= \binom{\ell+q-2}{\ell}\mathcal{P}_\ell(q+1; t) \quad (q \ge 2), \\ P_\ell^{(\frac{q}{2}-1, \frac{q}{2}-1)}(t) &= \binom{\ell+\frac{q}{2}-1}{\ell}\mathcal{P}_\ell(q+1; t). \end{aligned} \tag{2.2}$$

When $q = 1$, the Legendre polynomials $\mathcal{P}_\ell(2; \circ)$ coincide with the Chebyshev polynomials T_ℓ; the ultraspherical polynomials are then given by $P_\ell^{(0)}(t) = (2/\ell)T_\ell(t)$, if $\ell \ge 1$. For $\ell = 0$, $P_0^{(0)}(t) = 1$.

Let $\mathcal{S}$ be the space of all infinitely often differentiable functions on $\mathbb{S}^q$, endowed with the locally convex topology induced by the supremum norms of all the derivatives of such functions, and let $\mathcal{S}^*$ be the dual of this space, that is, the space of distributions on $\mathbb{S}^q$. For $x^* \in \mathcal{S}^*$, we define

$$\widehat{x^*}(\ell, k) := x^*(Y_{\ell,k}), \;\; k = 1, \cdots, d_\ell^q, \;\; \ell = 0, 1, \cdots. \tag{2.3}$$

A most common example is when x^* is defined by $g \mapsto \int_{\mathbb{S}^q} g(\xi)f(\xi)d\mu_q(\xi)$ for some integrable function f on $\mathbb{S}^q$. In this case, we identify x^* with f and use the notation $\hat{f}(\ell, k)$ to denote the corresponding $\widehat{x^*}(\ell, k)$.

2.2 Besov spaces

Our definition of Besov spaces is motivated by the equivalence theorem for the characterization of Besov spaces on the sphere ([13, Theorem 3.1]). For any integer $N \geq 0$, $1 \leq p \leq \infty$ and $f \in L^p$, we write

$$E_{N,p}(f) := \min_{P \in \Pi_N^q} \|f - P\|_p.$$

We will define the Besov spaces in terms of the sequence $\{E_{2^j,p}(f)\}_{j\in\mathbb{N}_0}$. Let $0 < \rho \leq \infty$, $\gamma > 0$, and let $\mathbf{a} = \{a_j\}_{j\in\mathbb{N}_0}$ be a sequence of real numbers. We define

$$\|\mathbf{a}\|_{\rho,\gamma} := \begin{cases} \left(\sum_{j=0}^{\infty} 2^{j\gamma\rho}|a_j|^\rho\right)^{1/\rho}, & \text{if } 0 < \rho < \infty, \\ \sup_{j\in\mathbb{N}_0} 2^{j\gamma}|a_j|, & \text{if } \rho = \infty. \end{cases}$$

The space of sequences $\mathbf{a}$ for which $\|\mathbf{a}\|_{\rho,\gamma} < \infty$ will be denoted by $\mathsf{b}_{\rho,\gamma}$. If $1 \leq p \leq \infty$, the Besov space $B^\gamma_{p,\rho}$ consists of all functions $f \in L^p$ for which $\{E_{2^j,p}(f)\}_{j\in\mathbb{N}_0} \in \mathsf{b}_{\rho,\gamma}$. The expression

$$\|f\|_{p,\rho,\gamma} := \begin{cases} \|f\|_p + \left(\sum_{j=0}^{\infty} \left[2^{j\gamma}\, E_{2^j,p}(f)\right]^\rho\right)^{1/\rho}, & \text{if } 0 < \rho < \infty, \\ \|f\|_p + \sup_{j\in\mathbb{N}_0} 2^{j\gamma} E_{2^j,p}(f), & \text{if } \rho = \infty, \end{cases} \tag{2.4}$$

defines a quasi–norm on the space, and the unit ball $\mathcal{B}^\gamma_{p,\rho} := \{f \;:\; \|f\|_{p,\rho,\gamma} \leq 1\}$ is a compact subset of L^p (cf. [15, Theorem 2.2]).

In the remainder of this paper, we adopt the following convention regarding constants. The letters $c, c_1, \cdots$ will denote positive constants depending only on such fixed parameters in the discussion as the dimension q, the different norms involved in the formula, and any other explicitly mentioned quantities. Their value will be different at different occurrences, even within the same formula. The expression $A \sim B$ will mean $cA \leq B \leq c_1 A$.

For the convenience of the reader, we summarize certain facts about the sequence spaces $\mathsf{b}_{\rho,\gamma}$ in the following lemma.

Lemma 1 *Let $0 < \rho \leq \infty$, $0 < \beta < \gamma$, $\{a_j\}_{j\in\mathbb{N}_0}$ be a sequence of real numbers.*
(a) *We have*

$$\|\{a_j\}_{j\in\mathbb{N}_0}\|_{\rho,\gamma} = \|\{2^{j\beta}a_j\}_{j\in\mathbb{N}_0}\|_{\rho,\gamma-\beta}. \tag{2.5}$$

(b) (Discrete Hardy inequality) *We have*

$$\left\|\left\{\sum_{j=n}^{\infty}|a_j|\right\}_{n\in\mathbb{N}_0}\right\|_{\rho,\gamma} \leq c\,\|\{a_j\}_{j\in\mathbb{N}_0}\|_{\rho,\gamma}. \tag{2.6}$$

Proof. Part (a) is clear from the definition. Part (b) is proved, for example, in [5, Lemma 3.4, p. 27]. □

Next, we illustrate a connection between Besov spaces and the smoothness conditions in [10, 11]. For $\gamma > 0$ and sufficiently smooth f (or more generally, for a distribution f associated with the coefficients $\hat{f}(\ell, k)$) we may define the fractional differentiation pseudo-differential operator ∂_γ by

$$\widehat{\partial_\gamma f}(\ell, k) := (\ell+1)^\gamma \hat{f}(\ell, k), \qquad \ell = 0, 1, \cdots, \quad k = 1, \cdots, d_\ell^q. \tag{2.7}$$

Proposition 2 *Let $\gamma > 0$, $f \in L^2$. Then $f \in B_{2,2}^\gamma$ if and only if $\partial_\gamma f \in L^2$, and if this holds then*

$$\|\partial_\gamma f\|_2 \sim \|f\|_{2,2,\gamma}.$$

PROOF. In view of the Parseval identity, we observe that for $f \in L^2$,

$$[E_{m,2}(f)]^2 = \sum_{\ell=m+1}^{\infty} \sum_{k=1}^{d_\ell^q} |\hat{f}(\ell, k)|^2, \qquad m = 0, 1, 2, \cdots.$$

Using the Parseval identity again,

$$\begin{aligned}
\|\partial_\gamma f\|_2^2 &= \sum_{\ell=0}^{\infty} \sum_{k=1}^{d_\ell^q} (\ell+1)^{2\gamma} |\hat{f}(\ell, k)|^2 \\
&= \sum_{\ell=0}^{1} \sum_{k=1}^{d_\ell^q} (\ell+1)^{2\gamma} |\hat{f}(\ell, k)|^2 + \sum_{j=0}^{\infty} \sum_{\ell=2^j+1}^{2^{j+1}} \sum_{k=1}^{d_\ell^q} (\ell+1)^{2\gamma} |\hat{f}(\ell, k)|^2 \\
&\sim \sum_{\ell=0}^{1} \sum_{k=1}^{d_\ell^q} |\hat{f}(\ell, k)|^2 + \sum_{j=0}^{\infty} 2^{2j\gamma} \sum_{\ell=2^j+1}^{2^{j+1}} \sum_{k=1}^{d_\ell^q} |\hat{f}(\ell, k)|^2 \\
&= \sum_{\ell=0}^{1} \sum_{k=1}^{d_\ell^q} |\hat{f}(\ell, k)|^2 + \sum_{j=0}^{\infty} 2^{2j\gamma} \left([E_{2^j,2}(f)]^2 - [E_{2^{j+1},2}(f)]^2 \right) \\
&= \sum_{\ell=0}^{1} \sum_{k=1}^{d_\ell^q} |\hat{f}(\ell, k)|^2 + \sum_{j=0}^{\infty} 2^{2j\gamma} [E_{2^j,2}(f)]^2 - 2^{-2\gamma} \sum_{j=1}^{\infty} 2^{2j\gamma} [E_{2^j,2}(f)]^2 \\
&\sim \sum_{\ell=0}^{1} \sum_{k=1}^{d_\ell^q} |\hat{f}(\ell, k)|^2 + \sum_{j=0}^{\infty} 2^{2j\gamma} [E_{2^j,2}(f)]^2.
\end{aligned}$$

The proposition now follows from the definitions. $\square$

Finally, we recall an alternative characterization of the Besov spaces using polynomial operators [15, 19, 4]. In the sequel, $h : [0, \infty) \to [0, \infty)$ will denote a fixed function with the following properties: (i) h is infinitely differentiable, (ii) h is nonincreasing, (iii) $h(x) = 1$ if $x \le 1/2$, and (iv) $h(x) = 0$ if $x \ge 1$. All the generic constants $c, c_1, \cdots$ may depend upon the choice of h. The univariate polynomials Φ_j of degree at most $2^j - 1$ are defined by

$$\Phi_j(h, t) := \sum_{\ell=0}^{\infty} h\left(\frac{\ell}{2^j}\right) \frac{d_\ell^q}{\omega_q} \mathcal{P}_\ell(q+1; t), \qquad t \in \mathbb{R}, \ j = 0, 1, \cdots. \tag{2.8}$$

We define $\Phi_j(h,t)=0$ if $j<0$. We will need the following operators defined for $f\in L^1$ and integer j: The summability operator σ_j is defined by

$$\begin{aligned}\sigma_j(f)(\mathbf{x}) &:= \int_{\mathbb{S}^q} f(\mathbf{y})\Phi_j(h,\mathbf{x}\cdot\mathbf{y})d\mu_q(\mathbf{y})\\ &= \sum_{\ell=0}^{\infty}\sum_{k=1}^{d_\ell^q} h\left(\frac{\ell}{2^j}\right)\hat{f}(\ell,k)Y_{\ell,k}(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{S}^q, \qquad (2.9)\end{aligned}$$

and the frame operator τ_j is defined by

$$\begin{aligned}\tau_j(f) &= \sigma_j(f)-\sigma_{j-1}(f)\\ &= \sum_{\ell=0}^{\infty}\sum_{k=1}^{d_\ell^q}\left(h\left(\frac{\ell}{2^j}\right)-h\left(\frac{\ell}{2^{j-1}}\right)\right)\hat{f}(\ell,k)Y_{\ell,k}. \qquad (2.10)\end{aligned}$$

Note that $\tau_j(f)$ and $\sigma_j(f)$ are polynomials of degree $\le 2^j-1$, and that $\tau_j(f)$ is L^2-orthogonal to all polynomials of degree $\le 2^{j-2}$.

The proof of the following theorem is sketched in the Appendix.

Theorem 3 *Let $1\le p\le\infty$, and let $f\in X^p$. Then the decomposition*

$$f=\sum_{j=0}^{\infty}\tau_j(f) \qquad (2.11)$$

holds in the sense of convergence in X^p. Furthermore, for $0<\rho\le\infty$ and $\gamma>0$,

$$c_1\|f\|_{p,\rho,\gamma}\le\|f\|_p+\left\|\{\|\tau_j(f)\|_p\}_{j\in\mathbb{N}_0}\right\|_{\rho,\gamma}\le c_2\|f\|_{p,\rho,\gamma}. \qquad (2.12)$$

In particular, $f\in B^\gamma_{p,\rho}$ if and only if $\{\|\tau_j(f)\|_p\}_{j\in\mathbb{N}_0}\in\mathsf{b}_{\rho,\gamma}$.

The following corollary of the above theorem will be used tacitly throughout the paper.

Corollary 4 *Let $1\le p\le\infty$, $\gamma>q/p$ and $0<\rho\le\infty$. If $f\in B^\gamma_{p,\rho}$ then f is almost everywhere equal to a continuous function on $\mathbb{S}^q$, the series in (2.11) converges uniformly to this continuous function, and $\|f\|_\infty\le c\|f\|_{p,\rho,\gamma}$.*

Proof. For $p=\infty$ nothing needs to be shown because X^∞ convergence of (2.11) implies uniform convergence, and (2.4) directly yields $\|f\|_\infty\le\|f\|_{\infty,\rho,\gamma}$. For $1\le p<\infty$, the Nikolskii inequality gives

$$\|P\|_\infty\le cN^{q/p}\|P\|_p, \qquad P\in\Pi_N^q$$

(for a proof see [18, Proposition 2.1]). Since $\tau_j(f)\in\Pi^q_{2^j}$, we have

$$\|\tau_j(f)\|_\infty\le c2^{jq/p}\|\tau_j(f)\|_p.$$

Because $f \in B^{\gamma}_{p,\rho}$, (2.12) in Theorem 3 implies, in particular, that $\|\tau_j(f)\|_p \le c2^{-j\gamma}\|f\|_{p,\rho,\gamma}$. Therefore, for $\gamma > q/p$

$$\sum_{j=0}^{\infty} \|\tau_j(f)\|_\infty \le c\|f\|_{p,\rho,\gamma} \sum_{j=0}^{\infty} 2^{-j(\gamma - q/p)} = c_1\|f\|_{p,\rho,\gamma}.$$

Thus, $\sum_{j=0}^{\infty} \tau_j(f)$ converges uniformly to a continuous function on $\mathbb{S}^q$. In view of the X^p convergence of (2.11), this continuous function is equal to f almost everywhere on $\mathbb{S}^q$. Moreover,

$$\|f\|_\infty \le \sum_{j=0}^{\infty} \|\tau_j(f)\|_\infty = c_1\|f\|_{p,\rho,\gamma}.$$

□

3 Main results

Our main purpose in this section is to describe the upper and lower bounds on quadrature formulas on the sphere. Although we are mainly interested at this time in the quadrature formulas of the form $\sum_{\mathbf{x}\in\mathcal{C}} w_{\mathbf{x}} f(\mathbf{x})$, where $\mathcal{C}$ is a finite subset of $\mathbb{S}^q$, we prefer to state our theorems for more general approximations of the integral $\int f d\mu_q$, for example, allowing approximations which involve averages of f on finitely many caps rather than point evaluations. Towards this goal, we observe that for any point $\mathbf{x} \in \mathbb{S}^q$, one has the Dirac–delta measure $\delta_{\mathbf{x}}$, with the property that for any continuous $f : \mathbb{S}^q \to \mathbb{R}$, we have

$$\int_{\mathbb{S}^q} f d\delta_{\mathbf{x}} = f(\mathbf{x}).$$

So, one may write

$$\sum_{\mathbf{x}\in\mathcal{C}} w_{\mathbf{x}} f(\mathbf{x}) = \int_{\mathbb{S}^q} f d\nu,$$

where the measure ν is defined by $\nu = \sum_{\mathbf{x}\in\mathcal{C}} w_{\mathbf{x}}\delta_{\mathbf{x}}$. This notation has the advantage that we need not specify the points $\mathbf{x}$, the weights $w_{\mathbf{x}}$, or the number of points involved in the sum. We recall that the total variation measure of any signed measure ν is defined by

$$|\nu|(\mathcal{U}) := \sup \sum_{i=1}^{\infty} |\nu(\mathcal{U}_i)|, \qquad \mathcal{U} \subset \mathbb{S}^q,$$

where the supremum is taken over all countable partitions $\{\mathcal{U}_i\}$ of $\mathcal{U}$. In the case when $\nu = \sum_{\mathbf{x}\in\mathcal{C}} w_{\mathbf{x}}\delta_{\mathbf{x}}$, one can easily deduce that $|\nu|(\mathbb{S}^q) = \sum_{\mathbf{x}\in\mathcal{C}} |w_{\mathbf{x}}|$.

A (closed) spherical cap with center $\mathbf{y}$ and (angular) radius α is defined by

$$\mathbb{S}^q_\alpha(\mathbf{y}) := \{\mathbf{x} \in \mathbb{S}^q \ : \ \mathrm{dist}(\mathbf{x}, \mathbf{y}) \le \alpha\},$$

where $\mathrm{dist}(\mathbf{x}, \mathbf{y})$ is the geodesic distance on $\mathbb{S}^q$,

$$\mathrm{dist}(\mathbf{x}, \mathbf{y}) := \cos^{-1}(\mathbf{x} \cdot \mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in \mathbb{S}^q.$$

(That the geodesic distance is a metric is very well known for $q = 2$. That the triangle inequality $\mathrm{dist}(\mathbf{x}, \mathbf{y}) \le \mathrm{dist}(\mathbf{x}, \mathbf{z}) + \mathrm{dist}(\mathbf{y}, \mathbf{z})$ holds for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{S}^q$ and general q follows from the fact that the intersection of $\mathbb{S}^q$ with span $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is isomorphic to $\mathbb{S}^2$, or to $\mathbb{S}^1$ or $\mathbb{S}^0$, for which the result is already known.)

If $r > 0$, $A \ge 1$, and ν is a (possibly signed) measure, we will say that ν is (A, r)–continuous if for every spherical cap C, we have

$$|\nu|(C) \le A(\mu_q(C) + r^q). \tag{3.1}$$

Note that μ_q is $(1, r)$-continuous for every $r > 0$, and so is any measure μ of the form $\mu(B) = \int_B f d\mu_q$ for fixed $f \in L^\infty$, $\|f\|_\infty \le 1$. In general, ν is (A, r)–continuous if and only if $|\nu|/A$ is $(1, r)$–continuous. In view of Lemma 9 below, (3.1) is equivalent to the statement that for some constant $A_1 > 0$,

$$|\nu|(C) \le A_1 \mu_q(C)$$

for all caps C of radius at least r. In particular, for fixed $A > 0$ and arbitrary r, with $0 < r \le 1$, every (A, r)–continuous measure ν satisfies

$$|\nu|(\mathbb{S}^q) \le c, \tag{3.2}$$

with c independent of ν (i.e. c may depend on A, but not on $r \in (0, 1]$).

We now formulate our upper bound on the quadrature error as follows.

Theorem 5 *Let $A \ge 1$, $1 \le p \le \infty$, $0 < \rho \le \infty$, and $\gamma > q/p$. For any sequence $\{\nu_N\}_{N \in \mathbb{N}_0}$ of (possibly signed) $(A, 2^{-N})$–continuous measures ν_N on $\mathbb{S}^q$ that satisfy*

$$\int_{\mathbb{S}^q} P d\nu_N = \int_{\mathbb{S}^q} P d\mu_q, \qquad P \in \Pi^q_{2^N},\ N \in \mathbb{N}_0, \tag{3.3}$$

the estimate

$$\left\| \left\{ \left| \int_{\mathbb{S}^q} f d\mu_q - \int_{\mathbb{S}^q} f d\nu_N \right| \right\}_{N \in \mathbb{N}_0} \right\|_{\rho, \gamma} \le c \|f\|_{p, \rho, \gamma}, \qquad f \in B^\gamma_{p, \rho}, \tag{3.4}$$

holds.

We observe that most of the constructions mentioned in the introduction yield positive measures ν_N, supported on $\mathcal{O}(2^{Nq})$ points of $\mathbb{S}^q$, such that (3.3) is satisfied. Reimer [24, Lemma 1] has proved that all such measures satisfy the continuity condition required in the theorem. A simpler proof, stated for arbitrary positive measures satisfying (3.3), is given in [16, Theorem 3.3].

The estimate (3.4) clearly implies that

$$\left| \int_{\mathbb{S}^q} f d\mu_q - \int_{\mathbb{S}^q} f d\nu_N \right| \le c 2^{-N\gamma} \|f\|_{p, \rho, \gamma}. \tag{3.5}$$

In view of Proposition 2, this estimate, with $p = \rho = 2$, implies that of Hesse and Sloan in [10] for $\mathbb{S}^2$ and also the extension to $\mathbb{S}^q$, $q \ge 2$, in [1].

We now turn our attention to the lower bounds. These will be obtained by constructing a "bad function" for each quadrature formula such that the quadrature error for this function is estimated from below by the right–hand side of (3.5), assuming $M = \mathcal{O}(2^{Nq})$.

Let Q be *any* quadrature formula based on M points. In [11, Theorem 1] for $\mathbb{S}^2$ and in [9, Theorem 1] for $\mathbb{S}^q$, with arbitrary $q \geq 2$, it was shown that there exists a constant c, independent of M and of the quadrature points and weights, such that there exist $2M$ disjoint caps on the sphere, each of radius $c/M^{1/q}$. Necessarily, there are at least M of these caps that do not contain any of the quadrature points. Then the "bad function" was constructed as a sum of judiciously chosen functions, each supported on one of the caps not containing a quadrature point.

In the present paper we prefer a variant of the argument, in which M spherical caps containing quadrature points are supposed given, and the need is to construct M new caps which are disjoint from each other and from the original caps. The advantage of this approach is that it can be extended to a somewhat more general setting.

For a compact set $K \subset \mathbb{S}^q$, and $\epsilon > 0$, let the ϵ–neighborhood of K be defined by

$$\mathcal{N}_\epsilon(K) := \{\mathbf{x} \in \mathbb{S}^q \ : \ \text{dist}\ (\mathbf{x}, K) \leq \epsilon\}.$$

If K consists of M points then $\mathcal{N}_\epsilon(K)$ is the union of M spherical caps of angular radius ϵ. In general, if K is the support of any signed measure, and if for some fixed $\delta \in (0,1)$ and sufficiently small $\epsilon > 0$ we have

$$\mu_q(\mathcal{N}_\epsilon(K)) \leq (1-\delta)\omega_q, \tag{3.6}$$

then we will show that there exist at least $c\epsilon^{-q}$ mutually disjoint caps of radius $\epsilon/2$ which are also disjoint from K. (The constant c may depend on δ.) Our lower bound will be stated in terms of ϵ, allowing us to choose the largest ϵ for which (3.6) is satisfied. In the case in which K is a well distributed configuration of M points, the largest such ϵ satisfies $\epsilon \sim M^{-1/q}$.

Theorem 6 *Let $1 \leq p \leq \infty$, $0 < \rho \leq \infty$, $\gamma > q/p$, let ν be any signed measure supported on a compact subset $K \subset \mathbb{S}^q$, and let $\delta \in (0,1)$ and $\epsilon > 0$ be such that*

$$\mu_q(\mathcal{N}_\epsilon(K)) \leq (1-\delta)\omega_q.$$

Then there exists $f^* \not\equiv 0$, $f^* \in \mathcal{B}^\gamma_{p,\rho}$, such that

$$\left| \int_{\mathbb{S}^q} f^* d\mu_q - \int_{\mathbb{S}^q} f^* d\nu \right| \geq c(\delta)\epsilon^\gamma \|f^*\|_{p,\rho,\gamma}, \tag{3.7}$$

where $c(\delta)$ is a positive constant depending only on δ, q, p, ρ, and γ, but not on ν, f^*, ϵ, or K. In particular, if for each integer $M \geq 1$, ν is a signed measure supported on at most M points, then

$$\sup_{f \in \mathcal{B}^\gamma_{p,\rho}} \left| \int_{\mathbb{S}^q} f d\mu_q - \int_{\mathbb{S}^q} f d\nu \right| \geq c_1 M^{-\gamma/q}, \tag{3.8}$$

with a positive constant c_1 depending on q, p, ρ, and γ, but not on ν.

We conclude this section with an explicit result for the worst–case complexity of quadrature based on finitely many points. This exploits the agreement with respect to order between the upper bound stated in (3.5) with $M \sim 2^{Nq}$ and the lower bound (3.8). If $M \geq 1$ is an integer, $\mathbf{w} = (w_1, \cdots, w_M)$, $\mathcal{C} = \{\mathbf{x}_1, \cdots, \mathbf{x}_M\} \subset \mathbb{S}^q$, we write for $\gamma > q/p$,

$$\mathrm{error}_{q,p,\rho,\gamma,M}(\mathbf{w}, \mathcal{C}) := \sup_{\|f\|_{p,\rho,\gamma}=1} \left| \int_{\mathbb{S}^q} f d\mu_q - \sum_{k=1}^{M} w_k f(\mathbf{x}_k) \right|,$$

and

$$\mathcal{E}_{q,p,\rho,\gamma,M} := \inf\{\mathrm{error}_{q,p,\rho,\gamma,M}(\mathbf{w}, \mathcal{C}) \ : \ \mathbf{w} \in \mathbb{R}^M, \ \mathcal{C} \subset \mathbb{S}^q, \ |\mathcal{C}| = M\}.$$

Theorem 7 *Let $1 \leq p \leq \infty$, $0 < \rho \leq \infty$, and $\gamma > q/p$. Then for $M \geq 1$,*

$$\mathcal{E}_{q,p,\rho,\gamma,M} \sim M^{-\gamma/q}.$$

4 Proofs

In order to prove Theorem 5, we find it convenient to introduce another kernel. We define

$$\tilde{\Phi}_j(h, t) := \Phi_{j+1}(h, t) - \Phi_{j-2}(h, t) = \sum_{\ell=0}^{\infty} \tilde{\mathbf{h}}_j(\ell) \frac{d_\ell^q}{\omega_q} \mathcal{P}_\ell(q+1; t), \qquad t \in \mathbb{R}, \ j = 2, 3, \cdots, \tag{4.1}$$

where

$$\tilde{\mathbf{h}}_j(\ell) = h\left(\frac{\ell}{2^{j+1}}\right) - h\left(\frac{\ell}{2^{j-2}}\right).$$

We define $\tilde{\Phi}_j(h, t) = \Phi_j(h, t)$, $j = 0, 1$, and $\tilde{\Phi}_j(h, t) = \Phi_j(h, t) = 0$ if $j < 0$. The following Lemma 8 is essentially contained in [16, Proposition 4.1].

Lemma 8 *Let $1 \leq p \leq \infty$, $A \geq 1$, ν be a (possibly signed) (A, r)–continuous measure on $\mathbb{S}^q$, and $1/2^{j+1} \leq r \leq 1$. Then there exists a constant c independent of A and r such that*

$$\left\| \int_{\mathbb{S}^q} |\tilde{\Phi}_j(h, \circ \cdot \mathbf{y})| d|\nu|(\mathbf{y}) \right\|_p \leq c(|\nu|(\mathbb{S}^q))^{1/p} \left(A(2^j r)^q\right)^{1/p'}, \qquad j = 0, 1, \cdots, \tag{4.2}$$

where $1/p + 1/p' = 1$, with the usual understanding if $p = 1$ or $p = \infty$.

Proof. In this proof only, for any sequence $\{a_n\}_{n\in\mathbb{N}_0}$, let

$$\Delta a_n := \Delta^1 a_n := a_{n+1} - a_n, \qquad \Delta^k a_n := \Delta(\Delta^{k-1} a_n), \quad k = 2, 3, \cdots,$$

To apply Proposition 4.1 in [16], we define $\mathbf{h}(\ell) := \tilde{\mathbf{h}}_j(\ell)$, where $\tilde{\mathbf{h}}_j(\ell)$ is defined above, and observe that

$$\mathbf{h}(\ell) = 0 \qquad \text{for } \ell \geq 2^{j+1} \text{ and } \Delta\mathbf{h}(\ell) = 0 \text{ for } \ell \leq 2^{j-4}.$$

This allows us to apply Proposition 4.1 in [16] with $D = 2^{j+1}$, $C_1 = 1/32$, and $C_2 = 1$. Choosing also $K = q+1$ and $\rho = r$, and replacing $\Psi(\mathbf{h}, \mathbf{x}\cdot\mathbf{y})$ by $\tilde{\Phi}_j(h, \mathbf{x}\cdot\mathbf{y})$, the estimate (4.2) in Proposition 4.1 in [16] gives

$$\int_{\mathbb{S}^q} |\tilde{\Phi}_j(h, \mathbf{x}\cdot\mathbf{y})| d|\nu|(\mathbf{y}) \le cA(2^{j+1}r)^q \sum_{i=1}^{q+1} \sum_{\ell=0}^{2^{j+1}} (\ell+1)^{i-1} |\Delta^i \mathbf{h}(\ell)|, \qquad \mathbf{x} \in \mathbb{S}^q. \tag{4.3}$$

A repeated application of the mean value theorem shows that

$$|\Delta^i \mathbf{h}(\ell)| \le c \max_{t\in\mathbb{R}} |\mathbf{h}^{(i)}(t)| \le c2^{-ji} \max_{x\in\mathbb{R}} |h^{(i)}(x)| \le c2^{-ji}, \qquad 1 \le i \le q+1,$$

from which it follows that

$$\int_{\mathbb{S}^q} |\tilde{\Phi}_j(h, \mathbf{x}\cdot\mathbf{y})| d|\nu|(\mathbf{y}) \le cA(2^j r)^q, \qquad \mathbf{x} \in \mathbb{S}^q. \tag{4.4}$$

This proves (4.2) for the case $p = \infty$.

To prove the result for the case $p = 1$, it is useful to note that as a special case of (4.4) we have

$$\int_{\mathbb{S}^q} |\tilde{\Phi}_j(h, \mathbf{x}\cdot\mathbf{y})| d\mu_q(\mathbf{y}) \le c, \qquad \mathbf{x} \in \mathbb{S}^q,$$

since for the Lebesgue surface measure μ_q the assumption of (A, r)–continuity is satisfied for $A = 1$ and every choice of r, thus we may choose $r = 1/2^{j+1}$. Applying the Fubini theorem, we now obtain

$$\left\| \int_{\mathbb{S}^q} |\tilde{\Phi}_j(h, \circ\cdot\mathbf{y})| d|\nu|(\mathbf{y}) \right\|_1 = \int_{\mathbb{S}^q} \left(\int_{\mathbb{S}^q} |\tilde{\Phi}_j(h, \mathbf{x}\cdot\mathbf{y})| d\mu_q(\mathbf{x}) \right) d|\nu|(\mathbf{y}) \le c|\nu|(\mathbb{S}^q),$$

proving (4.2) for the case $p = 1$. The result in (4.2) then follows for all values of p from the interpolation inequality

$$\|g\|_p \le \|g\|_1^{1/p} \|g\|_\infty^{1/p'}, \qquad g \in L_1,$$

which follows by applying Hölder's inequality to $\|g\|_p^p$. $\square$

For the convenience of the reader, a self–contained sketch of the proof of the key estimate (4.3), following [14] and [16], is given in the Appendix.

Proof of Theorem 5. From the definitions (2.9) and (2.10), we see that $\sum_{j=0}^N \tau_j(f) = \sigma_N(f)$. Hence, (2.11) can be written in the form

$$f = \sigma_N(f) + \sum_{j=N+1}^{\infty} \tau_j(f), \qquad N \in \mathbb{N}_0,$$

where (cf. Corollary 4) the series converges uniformly on $\mathbb{S}^q$. Since $\sigma_N(f) \in \Pi_{2^N}^q$, we have, from the assumption (3.3),

$$\int_{\mathbb{S}^q} f d\nu_N = \int_{\mathbb{S}^q} \sigma_N(f) d\nu_N + \sum_{j=N+1}^{\infty} \int_{\mathbb{S}^q} \tau_j(f) d\nu_N = \int_{\mathbb{S}^q} \sigma_N(f) d\mu_q + \sum_{j=N+1}^{\infty} \int_{\mathbb{S}^q} \tau_j(f) d\nu_N.$$

Since (2.9) shows that $\int_{\mathbb{S}^q} \sigma_N(f) d\mu_q = \int_{\mathbb{S}^q} f d\mu_q$, we obtain

$$\epsilon_N := \left| \int_{\mathbb{S}^q} f d\mu_q - \int_{\mathbb{S}^q} f d\nu_N \right| = \left| \sum_{j=N+1}^{\infty} \int_{\mathbb{S}^q} \tau_j(f) d\nu_N \right| \leq \sum_{j=N+1}^{\infty} \left| \int_{\mathbb{S}^q} \tau_j(f) d\nu_N \right|. \tag{4.5}$$

We will estimate each of the terms on the right–hand side above. In the sequel, we will assume that $N \geq 2$, since for $N = 0$ and $N = 1$ the bound (3.5) is trivial. Now, let $j \geq N+1$. Since from (2.10)

$$\widehat{\tau_j(f)}(\ell, k) = \left(h\left(\frac{\ell}{2^j}\right) - h\left(\frac{\ell}{2^{j-1}}\right) \right) \hat{f}(\ell, k),$$

it follows from the definition of h in Section 2.2 that $\widehat{\tau_j(f)}(\ell, k) = 0$ if $\ell \geq 2^j$ or $\ell \leq 2^{j-2}$. So $\tau_j(f)$ is orthogonal to $\Pi^q_{2^{j-2}}$. In particular,

$$\int_{\mathbb{S}^q} \tau_j(f)(\mathbf{y}) \Phi_{j-2}(h, \mathbf{x} \cdot \mathbf{y}) d\mu_q(\mathbf{y}) = 0, \qquad \mathbf{x} \in \mathbb{S}^q.$$

Similarly, using the fact that $h(\ell/2^{j+1}) = 1$ if $\ell \leq 2^j$, we deduce that

$$\int_{\mathbb{S}^q} \tau_j(f)(\mathbf{y}) \Phi_{j+1}(h, \mathbf{x} \cdot \mathbf{y}) d\mu_q(\mathbf{y}) = \tau_j(f)(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{S}^q.$$

Therefore, for $j \geq N+1$ and $\mathbf{x} \in \mathbb{S}^q$, with (4.1),

$$\begin{aligned}
\tau_j(f)(\mathbf{x}) &= \int_{\mathbb{S}^q} \tau_j(f)(\mathbf{y}) \Phi_{j+1}(h, \mathbf{x} \cdot \mathbf{y}) d\mu_q(\mathbf{y}) - \int_{\mathbb{S}^q} \tau_j(f)(\mathbf{y}) \Phi_{j-2}(h, \mathbf{x} \cdot \mathbf{y}) d\mu_q(\mathbf{y}) \\
&= \int_{\mathbb{S}^q} \tau_j(f)(\mathbf{y}) \tilde{\Phi}_j(h, \mathbf{x} \cdot \mathbf{y}) d\mu_q(\mathbf{y}).
\end{aligned}$$

In essence, this holds because $\tilde{\mathbf{h}}_j(\ell) = 1$ whenever $\widehat{\tau_j(f)}(\ell, k) \neq 0$. Using Fubini's theorem, we then obtain

$$\int_{\mathbb{S}^q} \tau_j(f)(\mathbf{x}) d\nu_N(\mathbf{x}) = \int_{\mathbb{S}^q} \tau_j(f)(\mathbf{y}) \left\{ \int_{\mathbb{S}^q} \tilde{\Phi}_j(h, \mathbf{x} \cdot \mathbf{y}) d\nu_N(\mathbf{x}) \right\} d\mu_q(\mathbf{y}).$$

Since each ν_N is $(A, 2^{-N})$–continuous, an application of Hölder's inequality and (4.2) with p' in place of p now shows that (on noting also (3.2))

$$\left| \int_{\mathbb{S}^q} \tau_j(f) d\nu_N \right| \leq \|\tau_j(f)\|_p \left\| \int_{\mathbb{S}^q} \tilde{\Phi}_j(h, \mathbf{x} \cdot \circ) d\nu_N(\mathbf{x}) \right\|_{p'} \leq c(2^{j-N})^{q/p} \|\tau_j(f)\|_p. \tag{4.6}$$

From (4.5) and (4.6), we conclude that

$$2^{Nq/p} \epsilon_N \leq c \sum_{j=N+1}^{\infty} 2^{jq/p} \|\tau_j(f)\|_p.$$

Using (2.5), this leads to

$$\|\{\epsilon_N\}_{N\in\mathbb{N}_0}\|_{\rho,\gamma} = \left\|\{2^{Nq/p}\epsilon_N\}_{N\in\mathbb{N}_0}\right\|_{\rho,\gamma-q/p} \le \left\|\left\{\sum_{j=N+1}^{\infty} 2^{jq/p}\|\tau_j(f)\|_p\right\}_{N\in\mathbb{N}_0}\right\|_{\rho,\gamma-q/p}. \quad (4.7)$$

Since $f\in B^{\gamma}_{p,\rho}$, we may use (2.6), (2.5), and (2.12) to obtain

$$\begin{aligned}\left\|\left\{\sum_{j=N+1}^{\infty} 2^{jq/p}\|\tau_j(f)\|_p\right\}_{N\in\mathbb{N}_0}\right\|_{\rho,\gamma-q/p} &\le c\|\{2^{jq/p}\|\tau_j(f)\|_p\}_{j\in\mathbb{N}_0}\|_{\rho,\gamma-q/p}\\ &= c\|\{\|\tau_j(f)\|_p\}_{j\in\mathbb{N}_0}\|_{\rho,\gamma} \le c_2\|f\|_{p,\rho,\gamma}. \quad (4.8)\end{aligned}$$

From (4.5), (4.7), and (4.8), we finally obtain the estimate (3.4). □

We find it convenient to organize our proof of Theorem 6 in a number of lemmas. The following simple lemma gives a useful estimate on the volume of spherical caps.

Lemma 9 *Let* $\mathbf{y}\in\mathbb{S}^q$ *and* $\alpha\in[0,\pi]$. *Then*

$$\mu_q(\mathbb{S}^q_\alpha(\mathbf{y})) \sim \alpha^q. \quad (4.9)$$

PROOF. Let $\mathbf{n}$ be the point $(0,\cdots,0,1)\in\mathbb{S}^q$, using the standard Cartesian coordinate system. In view of the rotational invariance of μ_q, $\mu_q(\mathbb{S}^q_\alpha(\mathbf{y})) = \mu_q(\mathbb{S}^q_\alpha(\mathbf{n}))$. Writing $\mathbf{x}=(\sin\theta\mathbf{x}',\cos\theta)\in\mathbb{S}^q$, $\mathbf{x}'\in\mathbb{S}^{q-1}$, $\theta\in[0,\pi]$, we observe that

$$\mathbb{S}^q_\alpha(\mathbf{n}) = \{\mathbf{x}\in\mathbb{S}^q \ : \ \mathbf{x}\cdot\mathbf{n}\ge\cos\alpha\} = \{(\sin\theta\mathbf{x}',\cos\theta) \ : \ \mathbf{x}'\in\mathbb{S}^{q-1},\ 0\le\theta\le\alpha\}.$$

It is now elementary to check, as in [20], that

$$\mu_q(\mathbb{S}^q_\alpha(\mathbf{n})) = \int_{\mathbb{S}^q_\alpha(\mathbf{n})} d\mu_q(\mathbf{x}) = \omega_{q-1}\int_0^\alpha \sin^{q-1}\theta d\theta.$$

First, let $\alpha\in[0,\pi/2]$. Then the estimates

$$\frac{2}{\pi}\theta \le \sin\theta \le \theta, \qquad \theta\in[0,\pi/2],$$

show that

$$\mu_q(\mathbb{S}^q_\alpha(\mathbf{n})) \sim \alpha^q.$$

If $\alpha\in[\pi/2,\pi]$, then clearly, $\omega_q/2 \le \mu_q(\mathbb{S}^q_\alpha(\mathbf{n})) \le \omega_q$, completing the proof. □

The next lemma will allow us to establish the existence of adequately many mutually disjoint spherical caps in the complement of the support K of the measure ν in Theorem 6. In our application of the following lemma, G plays the role of the complement of an ϵ–neighborhood of K with $\alpha=\epsilon/2$.

Lemma 10 *Let $0 < \alpha \le \pi$, and let $G \subset \mathbb{S}^q$ be a compact set with $\mu_q(G) > 0$. There exists a finite subset $Y \subset G$ with cardinality $|Y|$ satisfying*

$$c\mu_q(G)\alpha^{-q} \le |Y| \le c_1\alpha^{-q}\mu_q(\mathcal{N}_\alpha(G)), \tag{4.10}$$

such that the caps $\mathbb{S}^q_\alpha(\mathbf{y})$, $\mathbf{y} \in Y$, are mutually disjoint, and the constants c and c_1 are independent of α and G.

Proof. In this proof only, we say that a set $Y \subset G$ is 2α–distinguishable, if $\mathrm{dist}(\mathbf{x}, \mathbf{y}) > 2\alpha$ for all $\mathbf{x}, \mathbf{y} \in Y$, $\mathbf{x} \ne \mathbf{y}$. Since G is compact, such a set is necessarily finite. Let Y be a maximal set of 2α–distinguishable points in G. If $\mathbf{x} \in G$, and $\mathbf{x} \notin \cup_{\mathbf{y}\in Y}\mathbb{S}^q_{2\alpha}(\mathbf{y})$, then $Y \cup \{\mathbf{x}\}$ is a strictly larger set of 2α–distinguishable points, which contradicts the maximal property of Y. Therefore, $G \subset \cup_{\mathbf{y}\in Y}\mathbb{S}^q_{2\alpha}(\mathbf{y})$. In view of (4.9), we deduce that

$$\mu_q(G) \le \sum_{\mathbf{y}\in Y} \mu_q(\mathbb{S}^q_{2\alpha}(\mathbf{y})) \le c|Y|\alpha^q.$$

This proves the first inequality in (4.10). Since Y is a set of 2α–distinguishable points, the caps $\mathbb{S}^q_\alpha(\mathbf{y})$, $\mathbf{y} \in Y$ are mutually disjoint. Since $\cup_{\mathbf{y}\in Y}\mathbb{S}^q_\alpha(\mathbf{y}) \subset \mathcal{N}_\alpha(G)$, it follows from (4.9) that

$$|Y|\alpha^q \sim \sum_{\mathbf{y}\in Y} \mu_q(\mathbb{S}^q_\alpha(\mathbf{y})) = \mu_q\left(\cup_{\mathbf{y}\in Y}\mathbb{S}^q_\alpha(\mathbf{y})\right) \le \mu_q(\mathcal{N}_\alpha(G)).$$

This proves the second inequality in (4.10). □

The following corollary will be used in our proof of Theorem 6.

Corollary 11 *Let $\delta, \epsilon \in (0, 1)$, and let K be a compact subset of $\mathbb{S}^q$ satisfying $\mu_q(\mathcal{N}_\epsilon(K)) < (1-\delta)\omega_q$. Then the closure of the set $\mathbb{S}^q \setminus \mathcal{N}_\epsilon(K)$ contains a finite subset Y with $|Y| \sim \epsilon^{-q}$, such that the caps $\mathbb{S}^q_{\epsilon/2}(\mathbf{y})$, $\mathbf{y} \in Y$ are mutually disjoint, and do not intersect K. The implied constants in $|Y| \sim \epsilon^{-q}$ may depend on δ but not on ϵ or K.*

Proof. We use Lemma 10, with G being the closure of $\mathbb{S}^q \setminus \mathcal{N}_\epsilon(K)$, and $\alpha = \epsilon/2$. The condition $\mu_q(\mathcal{N}_\epsilon(K)) < (1-\delta)\omega_q$ ensures that $\mu_q(G) > \delta\omega_q > 0$, and the fact that $\alpha < \epsilon$ ensures that the caps $\mathbb{S}^q_\alpha(\mathbf{y})$ do not intersect K. □

In particular, if K consists of M points, then there exist M mutually disjoint caps of radius $c/M^{1/q}$ which do not contain any point of K. This follows from the corollary on choosing say $\delta = 1/2$ and $\epsilon = 2c/M^{1/q}$ with a suitable choice of c.

Next, we establish a lemma, following ideas in [9], that will help us to estimate the iterated Laplace–Beltrami operators applied to zonal functions. The Laplace–Beltrami operator Δ^* is defined in the distributional sense by

$$\widehat{\Delta^* f}(\ell, k) := -\ell(\ell + q - 1)\hat{f}(\ell, k), \qquad \ell = 0, 1, \cdots, \ k = 1, \cdots, d^q_\ell.$$

It is known (cf. [20]) that Δ^* is the angular part of the Laplacian on $\mathbb{R}^{q+1}$. In particular, it is a surface differential operator, and if f is twice continuously differentiable and $f(\mathbf{x}) = 0$ on an open subset of $\mathbb{S}^q$, then $\Delta^* f(\mathbf{x}) = 0$ on that subset.

A *zonal function* is a function of the form $\mathbf{x} \in \mathbb{S}^q \mapsto f(\mathbf{x}\cdot\mathbf{z})$, where $\mathbf{z} \in \mathbb{S}^q$ is fixed and $f : [-1,1] \to \mathbb{R}$ is an integrable function. We will need the following facts (cf. [20]). If f is a zonal function, then with $t = \mathbf{x}\cdot\mathbf{z}$ we obtain

$$\int_{\mathbb{S}^q} f(\mathbf{x}\cdot\mathbf{z})\, d\mu_q(\mathbf{x}) = \omega_{q-1} \int_{-1}^{1} f(t)\,(1-t^2)^{(q-2)/2}\, dt.$$

Further, if $\Psi : [-1,1] \to \mathbb{R}$ is twice differentiable, and

$$\mathcal{D}_q \Psi(t) := -qt\Psi'(t) + (1-t^2)\Psi''(t),$$

we obtain

$$\Delta^* \Psi(\circ\cdot\mathbf{z}) = (\mathcal{D}_q\Psi)(\circ\cdot\mathbf{z}). \tag{4.11}$$

In the remainder of this section, let $\mathcal{C}_+$ denote the space of all infinitely differentiable functions Ψ on $(-\infty, 1]$ such that $\Psi(t) = 0$ for all $t \le 0$.

Lemma 12 *For every integer $s \ge 1$, there exist linear differential operators $T_{j,s} : \mathcal{C}_+ \to \mathcal{C}_+$ of order $\le 2s$ with the following property. Let $0 < \eta < 1$, $\Psi \in \mathcal{C}_+$, and $g(t) := \Psi((t-\eta)/(1-\eta))$. Then*

$$\mathcal{D}_q^s g(t) = \sum_{j=0}^{s} (1-\eta)^{-j} (T_{j,s}\Psi)\left(\frac{t-\eta}{1-\eta}\right). \tag{4.12}$$

Proof. Writing $u = (t-\eta)/(1-\eta)$ and hence $t = \eta + (1-\eta)u = 1-(1-\eta)(1-u)$, we calculate that

$$1-t^2 = 2(1-t) - (1-t)^2 = (1-\eta)\left(2(1-u) - (1-\eta)(1-u)^2\right),$$

and hence,

$$\begin{aligned}
\mathcal{D}_q g(t) &= -q\frac{t}{1-\eta}\Psi'(u) + \frac{1-t^2}{(1-\eta)^2}\Psi''(u) \\
&= \frac{1}{1-\eta}\left(-q\Psi'(u) + 2(1-u)\Psi''(u)\right) + \left(q(1-u)\Psi'(u) - (1-u)^2\Psi''(u)\right) \\
&=: \frac{1}{1-\eta}(T_{1,1}\Psi)(u) + (T_{0,1}\Psi)(u).
\end{aligned}$$

This proves (4.12) for $s = 1$. Clearly, $T_{1,1}\Psi$ and $T_{0,1}\Psi$ are both in $\mathcal{C}_+$. The general statements follow easily by induction. $\square$

Corollary 13 *Let $\Psi \in \mathcal{C}_+$, $0 < \beta < \pi/2$, $\mathbf{y} \in \mathbb{S}^q$, and let $\phi_{\mathbf{y},\beta} : \mathbb{S}^q \to \mathbb{R}$ be defined by $\phi_{\mathbf{y},\beta}(\mathbf{x}) := \Psi((\mathbf{x}\cdot\mathbf{y} - \cos\beta)/(1-\cos\beta))$, $\mathbf{x} \in \mathbb{S}^q$. Then for integer $s \ge 1$,*

$$\|(\Delta^*)^s \phi_{\mathbf{y},\beta}\|_\infty \le c\beta^{-2s}, \tag{4.13}$$

where c may depend on Ψ.

PROOF. Since $\Psi(t)=0$ for all $t\le 0$, we see that $\phi_{\mathbf{y},\beta}(\mathbf{x})=0$ if $\mathbf{x}\notin\mathbb{S}^q_\beta(\mathbf{y})$. Moreover, in view of (4.11) and Lemma 12, we see that for all $\mathbf{x}\in\mathbb{S}^q$

$$|(\Delta^*)^s\phi_{\mathbf{y},\beta}(\mathbf{x})| = \left|\sum_{j=0}^s(1-\cos\beta)^{-j}(T_{j,s}\Psi)\left(\frac{\mathbf{x}\cdot\mathbf{y}-\cos\beta}{1-\cos\beta}\right)\right| \le c(s,\Psi)(1-\cos\beta)^{-s}.$$

Since $1-\cos\beta = 2(\sin(\beta/2))^2\sim\beta^2$, this proves (4.13). $\square$

Next, we construct the function f^* required in Theorem 6. We define

$$\Psi(t) := \begin{cases} 0, & \text{if } t\le 0,\\ \left(\int_0^{1/2}\exp\left(\frac{2}{u(2u-1)}\right)du\right)^{-1}\int_0^t\exp\left(\frac{2}{u(2u-1)}\right)du, & \text{if } 0<t<1/2,\\ 1, & \text{if } 1/2\le t\le 1.\end{cases}$$

It is clear that $\Psi\in\mathcal{C}_+$, and $0\le\Psi(t)\le 1$ for all $t\in(-\infty,1]$.

Next, with K,δ and ϵ as in Theorem 6 we take the set Y as in Corollary 11, choose $\beta=\epsilon/2$, and define

$$\phi_{\mathbf{y},\beta}(\mathbf{x}) := \Psi\left(\frac{\mathbf{x}\cdot\mathbf{y}-\cos\beta}{1-\cos\beta}\right), \qquad \mathbf{x}\in\mathbb{S}^q,$$

and then

$$f^* = f^*(\epsilon) := \sum_{\mathbf{y}\in Y}\phi_{\mathbf{y},\beta}.$$

The support of the function f^* is the union of the $|Y|$ spherical caps $\mathbb{S}^q_\beta(\mathbf{y})=\mathbb{S}^q_{\epsilon/2}(\mathbf{y})$, $\mathbf{y}\in Y$, and is a subset of $\mathbb{S}^q\setminus\mathcal{N}_\epsilon(K)$. We recall from Corollary 11 that $|Y|\sim\epsilon^{-q}$.

In the next lemma, we estimate the Besov space norm of f^*.

Lemma 14 *Let $1\le p\le\infty$, $0<\rho\le\infty$, $\epsilon>0$, and $\gamma>0$. With $f^*=f^*(\epsilon)$ constructed as above we have*

$$\|f^*\|_{p,\rho,\gamma}\le c\epsilon^{-\gamma}. \tag{4.14}$$

PROOF. In this proof only, let s be the smallest integer such that $2s>\gamma$. We will assume at first that $\rho<\infty$; the case $\rho=\infty$ is simpler. Recalling that each $\phi_{\mathbf{y},\beta}$ with $\beta=\epsilon/2$ is supported on $\mathbb{S}^q_{\epsilon/2}(\mathbf{y})$, and that these caps are disjoint, we see that

$$f^*(\mathbf{x}) = \begin{cases}\phi_{\mathbf{y},\beta}(\mathbf{x}), & \text{if } \mathbf{x}\in\mathbb{S}^q_{\epsilon/2}(\mathbf{y}) \text{ for some } \mathbf{y}\in Y,\\ 0, & \text{otherwise.}\end{cases} \tag{4.15}$$

Thus, we have $\|f^*\|_\infty=\|\phi_{\mathbf{y},\beta}\|_\infty\le 1$. If $1\le p<\infty$, then using (4.9) and $|Y|\sim\epsilon^{-q}$, we conclude that

$$\|f^*\|_p^p = \sum_{\mathbf{y}\in Y}\int_{\mathbb{S}^q}|\phi_{\mathbf{y},\beta}|^p d\mu_q = \sum_{\mathbf{y}\in Y}\int_{\mathbb{S}^q_{\epsilon/2}(\mathbf{y})}|\phi_{\mathbf{y},\beta}|^p d\mu_q \le |Y|\mu_q(\mathbb{S}^q_{\epsilon/2}(\mathbf{y}))\le c\epsilon^{-q}\epsilon^q = c.$$

Thus, for all p, $1\le p\le\infty$,

$$E_{2^j,p}(f^*)\le\|f^*\|_p\le c, \qquad j=0,1,2,\cdots. \tag{4.16}$$

So,

$$\sum_{2^j<\epsilon^{-1}} \left(2^{j\gamma}E_{2^j,p}(f^*)\right)^\rho \le c \sum_{2^j<\epsilon^{-1}} 2^{j\gamma\rho} \le c\epsilon^{-\gamma\rho}, \tag{4.17}$$

where the last inequality follows from the formula for the geometric sum.

Next, $(\Delta^*)^s$ being a surface differential operator, we see that the supports of $(\Delta^*)^s\phi_{\mathbf{y},\beta}$, $\mathbf{y}\in Y$, are also mutually disjoint. Therefore, analogously to (4.15), we have

$$(\Delta^*)^s f^*(\mathbf{x}) = \begin{cases} (\Delta^*)^s\phi_{\mathbf{y},\beta}(\mathbf{x}), & \text{if } \mathbf{x}\in\mathbb{S}^q_{\epsilon/2}(\mathbf{y}) \text{ for some } \mathbf{y}\in Y,\\ 0, & \text{otherwise.}\end{cases} \tag{4.18}$$

Using (4.13), we obtain $\|(\Delta^*)^s\Phi_{\mathbf{y},\beta}\|_\infty \le c\epsilon^{-2s}$, and if $1\le p<\infty$, then

$$\|(\Delta^*)^s f^*\|_p^p = \sum_{\mathbf{y}\in Y}\int_{\mathbb{S}^q_{\epsilon/2}(\mathbf{y})} |(\Delta^*)^s\phi_{\mathbf{y},\beta}(\mathbf{x})|^p d\mu_q(\mathbf{x}) \le c\epsilon^{-2sp}|Y|\mu_q(\mathbb{S}^q_{\epsilon/2}(\mathbf{y})) \le c\epsilon^{-2sp},$$

where we have used $|Y|\sim\epsilon^{-q}$ and $\mu_q(\mathbb{S}^q_{\epsilon/2}(\mathbf{y}))\sim\epsilon^q$ from (4.9).

Thus, $\|(\Delta^*)^s f^*\|_p \le c\epsilon^{-2s}$ for all p, $1\le p\le\infty$, where the estimate for $p=\infty$ follows directly from (4.18) and (4.13).

In view of a result of Pawelke [21, Satz 3.3], this implies that

$$E_{2^j,p}(f^*) \le c2^{-2js}\epsilon^{-2s}, \qquad j=0,1,2,\cdots. \tag{4.19}$$

Therefore,

$$\sum_{2^j\ge\epsilon^{-1}} \left(2^{j\gamma}E_{2^j,p}(f^*)\right)^\rho \le c\epsilon^{-2s\rho} \sum_{2^j\ge\epsilon^{-1}} 2^{-j\rho(2s-\gamma)} \le c\epsilon^{-\gamma\rho}, \tag{4.20}$$

where the last estimate follows from summing a geometric series.

The estimate (4.14) for $0<\rho<\infty$ now follows from the definition (2.4), and the estimates (4.16), (4.17), and (4.20).

For $\rho=\infty$ we have from (4.16), (4.19), and $\gamma<2s$

$$\begin{aligned}
\|f^*\|_{p,\rho,\gamma} &\le \|f^*\|_p + \sup_{j\in\mathbb{N}_0} 2^{j\gamma}E_{2^j,p}(f^*)\\
&\le c + \max_{2^j<\epsilon^{-1}}\left\{2^{j\gamma}E_{2^j,p}(f^*)\right\} + \sup_{2^j\ge\epsilon^{-1}}\left\{2^{j\gamma}E_{2^j,p}(f^*)\right\}\\
&\le c + c\epsilon^{-\gamma} + \sup_{2^j\ge\epsilon^{-1}}\left\{c2^{j\gamma}2^{-2js}\epsilon^{-2s}\right\}\\
&\le c + c\epsilon^{-\gamma} + c\epsilon^{-\gamma}\sup_{2^j\ge\epsilon^{-1}}\left(\frac{2^j}{\epsilon^{-1}}\right)^{\gamma-2s}\\
&\le c\epsilon^{-\gamma},
\end{aligned}$$

which yields (4.14) for $\rho=\infty$. □

Finally, we are in a position to prove Theorem 6.

Proof of Theorem 6. We observe that the support of f^* is disjoint from the support of ν. Therefore,

$$\int_{\mathbb{S}^q} f^* d\nu = 0. \tag{4.21}$$

For each $\mathbf{y} \in Y$, $\phi_{\mathbf{y},\beta}(\mathbf{x}) = 1$ if $\mathbf{x}\cdot\mathbf{y} - \cos\beta \ge (1-\cos\beta)/2$; i.e., if $\mathbf{x}\cdot\mathbf{y} \ge \cos^2(\beta/2)$. Since $\cos(\beta/2) \ge \cos^2(\beta/2)$, we deduce that $\phi_{\mathbf{y},\beta}(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathbb{S}^q_{\beta/2}(\mathbf{y})$. Therefore, the fact that $\phi_{\mathbf{y},\beta}(\mathbf{x}) \ge 0$ for all $\mathbf{x} \in \mathbb{S}^q$ together with (4.9) implies that

$$\int_{\mathbb{S}^q} \phi_{\mathbf{y},\beta}(\mathbf{x}) d\mu_q(\mathbf{x}) \ge \int_{\mathbb{S}^q_{\beta/2}(\mathbf{y})} d\mu_q \ge c(\beta/2)^q \ge c\beta^q \ge c\epsilon^q.$$

Therefore, in view of (4.21) and $|Y| \sim \epsilon^{-q}$, we conclude that

$$\begin{aligned}\left|\int_{\mathbb{S}^q} f^* d\mu_q - \int_{\mathbb{S}^q} f^* d\nu\right| &= \int_{\mathbb{S}^q} f^* d\mu_q \\ &= \sum_{\mathbf{y}\in Y} \int_{\mathbb{S}^q} \phi_{\mathbf{y},\beta} d\mu_q \ge c|Y|\epsilon^q \ge c. \qquad (4.22)\end{aligned}$$

Now, (4.14) implies that $\epsilon^\gamma \|f^*\|_{p,\rho,\gamma} \le c_1$. Therefore, (4.22) implies (3.7). To obtain (3.8) we observe that in the case of a measure supported at M points, we can choose $\epsilon \sim M^{-1/q}$ and $\delta = 1/2$. □

Proof of Theorem 7. In view of (3.8), we have

$$\mathcal{E}_{q,p,\rho,\gamma,M} \ge cM^{-\gamma/q}.$$

To prove the upper bound, let $N \ge 2$ be chosen so that $2^{Nq} < M \le 2^{(N+1)q}$, $n = 2^{N-2}$, and $n_1 = (4n+1)(2n)^{q-1} \le M$. In [2, Theorem 4], Brown, Dai, and Sheng have constructed points $\mathbf{x}_{k,n}$, $k = 1, \cdots, n_1$, and positive numbers $w_{k,n}$, $k = 1, \cdots, n_1$, such that

$$\sum_{k=1}^{n_1} w_{k,n} P(\mathbf{x}_{k,n}) = \int_{\mathbb{S}^q} P d\mu_q, \qquad P \in \Pi^q_{2^N}.$$

If $k = n_1 + 1, \cdots, M$, we set $w_{k,n} = 0$, and choose arbitrary points $\mathbf{x}_{k,n} \in \mathbb{S}^q$, distinct from each other and from the nodes defined in the previous equation. Then

$$\sum_{k=1}^{M} w_{k,n} P(\mathbf{x}_{k,n}) = \int_{\mathbb{S}^q} P d\mu_q, \qquad P \in \Pi^q_{2^N}.$$

In view of (3.5), we conclude that for any f with $\|f\|_{p,\rho,\gamma} = 1$, we have

$$\left|\int_{\mathbb{S}^q} f d\mu_q - \sum_{k=1}^{M} w_{k,n} f(\mathbf{x}_{k,n})\right| \le c2^{-N\gamma} \le cM^{-\gamma/q}.$$

This proves the required upper bound on $\mathcal{E}_{q,p,\rho,\gamma,M}$. □

A Appendix

For the convenience of the reader, we sketch in this Appendix the proofs of Theorem 3 and the key estimate (4.3).

A.1 Proof of (4.3).

It is convenient to prove (4.3) first in a somewhat more general form. Let H be a sequence with $H(\ell) = 0$ for all $\ell \geq D$ for some D and $\Delta H(\ell) = 0$ for $0 \leq \ell \leq cD$. We write

$$\Phi(H,t) := \sum_{\ell=0}^{\infty} H(\ell)\frac{d_\ell^q}{\omega_q}\mathcal{P}_\ell(q+1;t) = \frac{1}{\omega_q}\sum_{\ell=0}^{\infty} H(\ell)\frac{2\ell+q-1}{q-1}\frac{(q-1)_\ell}{(q/2)_\ell}P_\ell^{(\frac{q}{2}-1,\frac{q}{2}-1)}(t), \quad \text{(A.1)}$$

where the Pochhammer symbol $(a)_\ell$ is defined for $a \in \mathbb{R}$ and $\ell \in \mathbb{N}_0$ by

$$(a)_0 := 1, \qquad (a)_\ell := a(a+1)\cdots(a+\ell-2)(a+\ell-1).$$

We note that both $\Phi_j(h,t)$ and $\tilde{\Phi}_j(h,t)$ are special cases of the kernel Φ, obtained with $h(\ell/2^j)$ and $h(\ell/2^{j+1}) - h(\ell/2^{j-2})$ in place of $H(\ell)$ respectively. We will prove that for any (A,r)–continuous measure ν,

$$\int_{\mathbb{S}^q} |\Phi(H,\mathbf{x}\cdot\mathbf{y})| d|\nu|(\mathbf{y}) \leq c(1+(Dr)^q)\sum_{i=1}^{q+1}\sum_{\ell=0}^{\infty}(\ell+1)^{i-1}|\Delta^i H(\ell)|, \qquad \mathbf{x}\in\mathbb{S}^q. \quad \text{(A.2)}$$

There are two major steps in this proof. First, we sketch the proof of an estimate on $|\Phi(H,t)|$ (see (A.9)), as given in [14]. Next, we use (A.9) to prove (A.2) as in [16].

During this proof only, let

$$\begin{aligned} b_\ell^{(0)} &:= H(\ell), \qquad b_\ell^{(m)} := \frac{q+m-1}{2\ell+q+m-1}\Delta b_\ell^{(m-1)}, \quad m\in\mathbb{N},\ \ell\in\mathbb{N}_0, \qquad \text{(A.3)}\\ a_\ell^{(m)} &:= \frac{2\ell+q+m-1}{q+m-1}\,\frac{(q+m-1)_\ell}{(q/2)_\ell}\,P_\ell^{(\frac{q}{2}-1+m,\frac{q}{2}-1)}(t), \quad m\in\mathbb{N}_0,\ \ell\in\mathbb{N}_0, \end{aligned}$$

As a simple consequence of the Christoffel-Darboux formula (see [28, (4.5.3)]) the Jacobi polynomials $\{P_\ell^{(\frac{q}{2}-1+m,\frac{q}{2}-1)}\}_{\ell\in\mathbb{N}_0}$, where $m \in \mathbb{N}_0$ is fixed, satisfy for all $\ell \in \mathbb{N}_0$

$$\sum_{r=0}^{\ell}\frac{2r+q+m-1}{q+m-1}\,\frac{(q+m-1)_r}{(q/2)_r}\,P_r^{(\frac{q}{2}-1+m,\frac{q}{2}-1)}(t) = \frac{(q+m)_\ell}{(q/2)_\ell}\,P_\ell^{(\frac{q}{2}+m,\frac{q}{2}-1)}(t). \quad \text{(A.4)}$$

Now we use the summation by parts formula

$$\sum_{\ell=0}^{\infty} a_\ell\, b_\ell = -\sum_{\ell=0}^{\infty}(b_{\ell+1}-b_\ell)\sum_{r=0}^{\ell} a_r$$

together with (A.4) to show

$$\sum_{\ell=0}^{\infty} a_\ell^{(m-1)} b_\ell^{(m-1)} = -\sum_{\ell=0}^{\infty} a_\ell^{(m)} b_\ell^{(m)}, \qquad m\in\mathbb{N},$$

and by repeating this $q+1$ times find

$$\Phi(H,t) = (-1)^{q+1}\frac{1}{\omega_q}\sum_{\ell=0}^{\infty} b_\ell^{(q+1)}\,\frac{2\ell+2q}{2q}\,\frac{(2q)_\ell}{(q/2)_\ell}\,P_\ell^{(\frac{3q}{2},\frac{q}{2}-1)}(t). \quad \text{(A.5)}$$

It can be shown by induction on m that the forward divided difference $\Delta b_\ell^{(m)}$ in the definition of $b_\ell^{(m+1)}$ has an expansion of the form (see [14, (4.12) in the proof of Lemma 4.4] for the details)

$$\Delta b_\ell^{(m)} = \sum_{k=1}^{m+1} \mathcal{R}_{m+1,k}(\ell+1)\, \Delta^k H(\ell), \qquad \ell \in \mathbb{N}_0,$$

where $\mathcal{R}_{m+1,k}$, $k \in \{1, \cdots, m+1\}$ is a rational function whose denominator has no non-negative zeros and a degree at least $2m-k+1$ higher than the degree of the numerator. Thus, with a positive constant c independent of ℓ, we find for $m = q$

$$|\Delta b_\ell^{(q)}| \leq c \sum_{k=1}^{q+1} (\ell+1)^{-2q+k-1}\, |\Delta^k H(\ell)|, \qquad \ell \in \mathbb{N}_0. \tag{A.6}$$

Using (A.5), (A.3), and (A.6), we obtain

$$|\Phi(H,t)| \leq c \sum_{k=1}^{q+1} \sum_{\ell=0}^{\infty} (\ell+1)^{-2q+k-1}\, |\Delta^k H(\ell)|\, \frac{(2q)_\ell}{(q/2)_\ell} \left| P_\ell^{(\frac{3q}{2},\frac{q-2}{2})}(t) \right|. \tag{A.7}$$

To estimate the kernel further we use the fact that (see [28, Theorem 7.32.2 and (4.1.3)])

$$\left| P_\ell^{(\frac{3q}{2},\frac{q}{2}-1)}(\cos\theta) \right| \leq c \begin{cases} (\ell+1)^{\frac{3q}{2}}, & \text{if } 0 \leq \theta \leq \frac{\pi}{2}, \\ (\ell+1)^{-\frac{1}{2}}\, \theta^{-\frac{3q+1}{2}}, & \text{if } \tilde{c}\,\ell^{-1} \leq \theta \leq \frac{\pi}{2}, \\ (\ell+1)^{\frac{q-2}{2}}, & \text{if } \frac{\pi}{2} \leq \theta \leq \pi. \end{cases} \tag{A.8}$$

The kth forward difference $\Delta^k H(\ell)$ is equal to 0 for $\ell \geq D$ and for $0 \leq \ell \leq cD-k+1$, and the estimate in the middle line in (A.8) is valid for $\hat{c}\, D^{-1} \leq \theta \leq \frac{\pi}{2}$ with a constant $\hat{c}$ independent of ℓ for $cD-q+1 < \ell < D$. Thus, (A.7), (A.8), and the fact that $(2q)_\ell/(q/2)_\ell \leq c\,(\ell+1)^{\frac{3q}{2}}$ imply the following estimates for the kernel $\Phi(H,t)$:

$$|\Phi(H,\cos\theta)| \leq c \begin{cases} \sum_{k=1}^{q+1} \sum_{\ell=0}^{D} (\ell+1)^{k+q-1}\, |\Delta^k H(\ell)|, & \text{if } 0 \leq \theta \leq \frac{\pi}{2}, \\ \sum_{k=1}^{q+1} \sum_{\ell=0}^{D} (\ell+1)^{k-\frac{q+3}{2}}\, |\Delta^k H(\ell)|\, \theta^{-\frac{3q+1}{2}}, & \text{if } \hat{c}\, D^{-1} \leq \theta \leq \frac{\pi}{2}, \\ \sum_{k=1}^{q+1} \sum_{\ell=0}^{D} (\ell+1)^{k-2}\, |\Delta^k H(\ell)|, & \text{if } \frac{\pi}{2} \leq \theta \leq \pi. \end{cases} \tag{A.9}$$

To estimate $\int_{\mathbb{S}^q} |\Phi(H,\mathbf{x}\cdot\mathbf{y})|\, d|\nu|(\mathbf{y})$ we split the sphere with respect to $\mathbf{x}$ as the north pole into three parts: the spherical cap $\mathbb{S}^q_{\hat{c}D^{-1}}(\mathbf{x})$, the rest of the northern hemisphere $\mathbb{S}^q_{\pi/2}(\mathbf{x}) \setminus \mathbb{S}^q_{\hat{c}D^{-1}}(\mathbf{x})$, and the southern hemisphere $\mathbb{S}^q_{\pi/2}(-\mathbf{x})$, and apply the corresponding estimates from (A.9). From the first estimate in (A.9) and $(\ell+1)^q \leq D^q$ for $\ell \leq D-1$ (note that $\Delta^k H(\ell) = 0$ for $\ell = D$), we obtain

$$\int_{\mathbb{S}^q_{\hat{c}D^{-1}}(\mathbf{x})} |\Phi(H,\mathbf{x}\cdot\mathbf{y})|\, d|\nu|(\mathbf{y}) \leq c\, D^q \sum_{k=1}^{q+1} \sum_{\ell=0}^{D} (\ell+1)^{k-1}\, |\Delta^k H(\ell)|\ |\nu|(\mathbb{S}^q_{\hat{c}D^{-1}}(\mathbf{x}))$$

$$
\begin{aligned}
&\le \; cD^q \sum_{k=1}^{q+1} \sum_{\ell=0}^{D} (\ell+1)^{k-1} \, |\Delta^k H(\ell)| \, A \left(\mu_q(\mathbb{S}^q_{\hat{c}D^{-1}}(\mathbf{x})) + r^q\right) \\
&\le \; c\,(1+(Dr)^q) \sum_{k=1}^{q+1} \sum_{\ell=0}^{D} (\ell+1)^{k-1} \, |\Delta^k H(\ell)|, \qquad \text{(A.10)}
\end{aligned}
$$

where we have used the (A,r)–continuity of the measure ν and (4.9). For the southern hemisphere $\mathbb{S}^q_{\pi/2}(-\mathbf{x})$, we find from the (A,r)–continuity of ν that $|\nu|(\mathbb{S}^q_{\pi/2}(-\mathbf{x})) \le c$. Thus, from the third line in (A.9),

$$
\begin{aligned}
\int_{\mathbb{S}^q_{\pi/2}(-\mathbf{x})} |\Phi(H, \mathbf{x}\cdot\mathbf{y})| \, d|\nu|(\mathbf{y}) \; &\le \; c \sum_{k=1}^{q+1} \sum_{\ell=0}^{D} (\ell+1)^{k-2} \, |\Delta^k H(\ell)| \; |\nu|(\mathbb{S}^q_{\pi/2}(-\mathbf{x})) \\
&\le \; c \sum_{k=1}^{q+1} \sum_{\ell=0}^{D} (\ell+1)^{k-1} \, |\Delta^k H(\ell)|. \qquad \text{(A.11)}
\end{aligned}
$$

Since $\Delta H(\ell) = 0$ for $0 \le \ell \le cD$ and for $\ell \ge D$, we obtain from the middle estimate in (A.9)

$$
\begin{aligned}
&\int_{\mathbb{S}^q_{\pi/2}(\mathbf{x}) \setminus \mathbb{S}^q_{\hat{c}D^{-1}}(\mathbf{x})} |\Phi(H, \mathbf{x}\cdot\mathbf{y})| \, d|\nu|(\mathbf{y}) \\
&\le \; c\, D^{-\frac{q+1}{2}} \sum_{k=1}^{q+1} \sum_{\ell=0}^{D} (\ell+1)^{k-1} \, |\Delta^k H(\ell)| \int_{\mathbb{S}^q_{\pi/2}(\mathbf{x}) \setminus \mathbb{S}^q_{\hat{c}D^{-1}}(\mathbf{x})} \theta^{-\frac{3q+1}{2}} \, d|\nu|(\mathbf{y}), \qquad \text{(A.12)}
\end{aligned}
$$

where $\cos\theta = \mathbf{x}\cdot\mathbf{y}$, $\theta \in [0,\pi]$. To evaluate the integral we write $W(\theta) := |\nu|(\mathbb{S}^q_\theta(\mathbf{x}))$, $\theta \in [0, \pi/2]$, and express the integral as a Riemann-Stieltjes integral

$$
\int_{\mathbb{S}^q_{\pi/2}(\mathbf{x}) \setminus \mathbb{S}^q_{\hat{c}D^{-1}}(\mathbf{x})} \theta^{-\frac{3q+1}{2}} \, d|\nu|(\mathbf{y}) = \int_{\hat{c}D^{-1}}^{\pi/2} \theta^{-\frac{3q+1}{2}} \, dW(\theta). \qquad \text{(A.13)}
$$

From the (A,r)–continuity of ν and (4.9),

$$
W(\theta) \le A \; (\mu_q(\mathbb{S}^q_\theta(\mathbf{x})) + r^q) \le A\,(c\,\theta^q + r^q)\,,
$$

and integration by parts yields

$$
\begin{aligned}
\int_{\hat{c}D^{-1}}^{\pi/2} \theta^{-\frac{3q+1}{2}} \, dW(\theta) \; &= \; \left[W(\theta)\, \theta^{-\frac{3q+1}{2}} \right]_{\hat{c}D^{-1}}^{\frac{\pi}{2}} + \frac{3q+1}{2} \int_{\hat{c}D^{-1}}^{\pi/2} W(\theta)\, \theta^{-\frac{3q+3}{2}} \, d\theta \\
&\le \; \tilde{c}\; (1+(Dr)^q) \; D^{\frac{q+1}{2}}.
\end{aligned}
$$

Substituting this estimate into (A.12) (see also (A.13)) yields

$$
\int_{\mathbb{S}^q_{\pi/2}(\mathbf{x}) \setminus \mathbb{S}^q_{\hat{c}D^{-1}}(\mathbf{x})} |\Phi(H, \mathbf{x}\cdot\mathbf{y})| \, d|\nu|(\mathbf{y}) \le c\; (1+(Dr)^q) \sum_{k=1}^{q+1} \sum_{\ell=0}^{D} (\ell+1)^{k-1} \, |\Delta^k H(\ell)|. \qquad \text{(A.14)}
$$

The estimate (A.2) follows from (A.10), (A.11), and (A.14).

Finally, we observe that $\tilde{\Phi}_j(h,t) = \Phi(H,t)$ with the choice $H(\ell) = h(\ell/2^{j+1}) - h(\ell/2^{j-2})$. In this case, $D = 2^{j+1}$, and our assumption on ν implies $Dr \ge c$. Thus, (A.2) with these choices implies (4.3). □

A.2 Proof of Theorem 3.

The proof of this Theorem is based on the following well known proposition.

Proposition 15 *Let $j \ge 0$ be an integer. For any $P \in \Pi^q_{2^{j-1}}$, $\sigma_j(P) = P$. If $f \in L^1$, then $\sigma_j(f) \in \Pi^q_{2^j}$. If $1 \le p \le \infty$ and $f \in L^p$, then $\|\sigma_j(f)\|_p \le c\|f\|_p$, and*

$$E_{2^j,p}(f) \le \|f - \sigma_j(f)\|_p \le cE_{2^{j-1},p}(f). \tag{A.15}$$

Although the facts contained in this proposition appear to be well known, we are not able to ascertain the original reference. Some proofs are given in [15, Proposition 4.1 and (4.16)] and [3, Lemma 2.5]. For the sake of completeness, we reproduce the proof from [15].

Proof of Proposition 15. The first two statements of the Proposition are clear from the relevant definitions. We observe that the measure μ_q is $(1, 1/D)$–continuous for every $D > 0$. We use (A.2) with the choice $H(\ell) = h(\ell/2^j)$, so that $D = 2^j$, and μ_q in place of ν, $1/D$ in place of r. With these choices, $\Phi(H, t) = \Phi_j(h, t)$, and we obtain

$$\int_{\mathbb{S}^q} |\Phi_j(h, \mathbf{x}\cdot\mathbf{y})| d\mu_q(\mathbf{y}) \le c \sum_{i=1}^{q+1} \sum_{\ell=0}^{\infty} (\ell + 1)^{i-1} |\Delta^i H(\ell)|, \qquad \mathbf{x} \in \mathbb{S}^q.$$

A repeated application of the mean value theorem yields, as in the proof of (4.4), that the right hand side the above inequality is bounded uniformly in j. Thus,

$$\int_{\mathbb{S}^q} |\Phi_j(h, \mathbf{x}\cdot\mathbf{y})| d\mu_q(\mathbf{y}) \le c, \qquad \mathbf{x} \in \mathbb{S}^q. \tag{A.16}$$

An application of Fubini's theorem now yields $\|\sigma_j(f)\|_1 \le c\|f\|_1$ for all $f \in L^1$ and $\|\sigma_j(f)\|_\infty \le c\|f\|_\infty$ for all $f \in L^\infty$. In view of the Riesz–Thorin interpolation theorem [5, Theorem 4.3], this yields $\|\sigma_j(f)\|_p \le c\|f\|_p$ for every $f \in L^p$. Since $\sigma_j(f) \in \Pi^q_{2^j}$ and $\sigma_j(P) = P$ for every $P \in \Pi^q_{2^{j-1}}$, we have for every $P \in \Pi^q_{2^{j-1}}$

$$E_{2^j,p}(f) \le \|f - \sigma_j(f)\|_p = \|f - P - \sigma_j(f - P)\|_p \le c\|f - P\|_p.$$

This completes the proof of (A.15). □

Proof of Theorem 3. This proof is based on the proof of the equivalence of parts (a) and (b) of [15, Theorem 3.3] (cf. also the proof of [19, Theorem 4]). From the definitions, we have for any integer $n \ge 0$, $\sigma_n(f) = \sum_{j=0}^n \tau_j(f)$. Therefore, (A.15) implies that

$$\left\| f - \sum_{j=0}^n \tau_j(f) \right\|_p = \|f - \sigma_n(f)\|_p \le cE_{2^{n-1},p}(f).$$

If $f \in X^p$ then $E_{2^{n-1},p}(f) \to 0$ as $n \to \infty$. This proves (2.11).

The series representation (2.11) and the estimates (A.15) imply that for $j = 0, 1, \cdots$,

$$E_{2^j,p}(f) \le \|f - \sigma_j(f)\|_p = \left\| f - \sum_{n=0}^j \tau_n(f) \right\|_p \le \sum_{n=j+1}^{\infty} \|\tau_n(f)\|_p.$$

Therefore, the discrete Hardy inequality (2.6) leads to the first inequality in (2.12).

Using (A.15) again (and recalling our convention that $\sigma_n = 0$ when $n < 0$), we see that for $n = 0, 1, 2, \cdots$,

$$\|\tau_n(f)\|_p = \|f - \sigma_n(f) - (f - \sigma_{n-1}(f))\|_p \le \|f - \sigma_n(f)\|_p + \|f - \sigma_{n-1}(f)\|_p \le cE_{2^{n-2},p}(f).$$

This implies the second inequality in (2.12). □

References

[1] J. S. Brauchart and K. Hesse, *Numerical integration over spheres of arbitrary dimension,* To appear in Constr. Approx..

[2] G. Brown, F. Dai, and Y. S. Sun, *Kolmogorov widths of classes of smooth functions on the sphere S^{d-1},* J. Complexity, **18** (2002), 1001–1023.

[3] F. Dai, *Jackson–type inequality for doubling weights on the sphere,* Const. Approx., **26** (2006), no. 1, 91–112.

[4] F. Dai, *Characterizations of function spaces on the sphere using frames,* To appear in Trans. Amer. Math. Soc..

[5] R. A. DeVore and G. G. Lorentz, "Constructive Approximation", Springer Verlag, Berlin, 1993.

[6] J. R. Driscoll and D. M. Healy, *Computing Fourier transforms and convolutions on the 2-sphere,* Adv. in Applied Math. **15**, 202-250 (1994).

[7] A. Erdélyi (ed.), W. Magnus, F. Oberhettinger and F. G. Tricomi (research associates), "Higher Transcendental Functions, Volume II", California Institute of Technology, Bateman Manuscript Project, McGraw-Hill Book Company, Inc., New York, Toronto, London, 1953.

[8] Q. T. Le Gia and H. N. Mhaskar, *Polynomial operators and local approximation of solutions of pseudo-differential equations on the sphere,* Numer. Math., **103** (2006), 299–322.

[9] K. Hesse, *A lower bound for the worst-case cubature error on spheres of arbitrary dimension,* To appear in Numer. Math..

[10] K. Hesse and I. H. Sloan, *Cubature over the sphere S^2 in Sobolev spaces of arbitrary order,* To appear in J. Approx. Theory.

[11] K. Hesse and I. H. Sloan, *Optimal lower bounds for cubature error on the sphere S^2,* J. Complexity, **21** (2005), 790–803.

[12] K. Jetter, J. Stöckler, and J. D. Ward, *Norming sets and spherical cubature formulas,* pp. 237-245 in "Computational Mathematics," (Chen, Li, C. Micchelli, Y. Xu, eds.), Marcel Decker, New York, 1998.

[13] P. I. Lizorkin and Kh. P. Rustamov, *Nikolskii–Besov spaces on the sphere in connection with approximation theory,* Tr. Mat. Inst. Steklova, **204** (1993), 172–201 (Proc. Steklov Inst. Math., **3** (1994), 149–172).

[14] H. N. Mhaskar, *Polynomial operators and local smoothness classes on the unit interval,* J. Approx. Theory, **131** (2004), 243-267.

[15] H. N. Mhaskar, *On the representation of smooth functions on the sphere using finitely many bits,* Appl. Comput. Harmon. Anal., **18** (3) (2005), 215-233.

[16] H. N. Mhaskar, *Weighted quadrature formulas and approximation by zonal function networks on the sphere,* J. Complexity, **22** (2006), 348–370.

[17] H. N. Mhaskar, F. J. Narcowich and J. D. Ward, *Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature,* Math. Comp. **70** (2001), no. 235, 1113–1130. (Corrigendum: Math. Comp. **71** (2001), 453–454.)

[18] H. N. Mhaskar, F. J. Narcowich, and J. D. Ward, *Approximation Properties of Zonal Function Networks Using Scattered Data on the Sphere,* Adv. Comput. Math., **11** (1999), 121-137.

[19] H. N. Mhaskar and J. Prestin, *Polynomial frames: a fast tour,* in "Approximation Theory, XI, Gatlinburg, 2004" (C. K. Chui, M. Neamtu, L. L. Schumaker eds.), Nashboro Press, Brentwood, 2005, pp. 287–318.

[20] C. Müller, "Spherical Harmonics", Lecture Notes in Mathematics, Vol. 17, Springer Verlag, Berlin, 1966.

[21] S. Pawelke, *Über die Approximationsordnung bei Kugelfunktionen und algebraischen Polynomen,* Tôhoku Math. Journ., **24** (1972), 473–486.

[22] D. Potts, G. Steidl, and M. Tasche, *Fast algorithms for discrete polynomial transforms,* Math. Comp., **67** (1998), 1577–1590.

[23] D.L. Ragozin, *Constructive polynomial approximation on sphere and projective spaces,* Trans. Amer. Math. Soc., **162** (1971), 157–170.

[24] M. Reimer, *Hyperinterpolation on the sphere at the minimal projection order,* J. Approx. Theory, **104** (2) (2000), 272–286.

[25] I.H. Sloan and R.S. Womersley, *Extremal systems of points and numerical integration on the sphere,* Adv. Comput. Math., **21** (2004), 107–125.

[26] E. M. Stein and G. Weiss, "Fourier Analysis on Euclidean Spaces", Princeton University Press, Princeton, New Jersey, 1971.

[27] A. H. Stroud, "Approximate Calculation of Multiple Integrals", Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971.

[28] G. Szegö, "Orthogonal Polynomials", Amer. Math. Soc. Colloq. Publ. Vol. 23, Amer. Math. Soc., Providence, 1975.