MATH 208, Midterm

Directions

1. Read the questions carefully, only answer what's asked from you.
2. Show your work. No work = no credit.
3. Academic dishonesty will not be tolerated in any form.

PROBLEMS:

1. Check if each of the following sequences converges or diverges. If it converges, find its limit as $n \to \infty$.

2 a) $\dfrac{(-1)^n n}{n+2}$ $\quad \dfrac{(-1)^n n}{n+2} = \dfrac{(-1)^n}{1+\frac{2}{n}} \xrightarrow[n\to\infty]{} (-1)^n$, so diverges

2 b) $\dfrac{(-1)^n n}{n^2+2}$ $\quad = \dfrac{(-1)^n}{n+\frac{2}{n}} \to 0$, so converges

2 c) $n^2 e^{-n}$ $\quad = \dfrac{n^2}{e^n} \xrightarrow[L'H]{} \dfrac{2n}{e^n} \xrightarrow[L'H]{} \dfrac{2}{e^n} = 0$, so converges

2. Check if each of the following series converges or diverges. Clearly state the tests you're using.

3 a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \cos n$, $\quad a_n = (-1)^n \cos n \not\to 0$, so Σ diverges

3 b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2-2}{n^3}$; $\quad a_n = \dfrac{n^2-2}{n^3}$, let $b_n = \frac{1}{n}$, then

$\dfrac{a_n}{b_n} = \dfrac{n^2-2}{n^3} \cdot n = \dfrac{n^2-2}{n^2} \to 1 \neq 0, \neq \infty$

so since $\Sigma \frac{1}{n}$ diverges,

$\Sigma \dfrac{n^2-2}{n^3}$ diverges by LCT

3 c) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$

Method 1:

$\ln n \geq 1$ for $n \geq 3$

so $\dfrac{\ln n}{n} \geq \dfrac{1}{n}$

& since $\Sigma \frac{1}{n}$ diverges, so does $\Sigma \frac{\ln n}{n}$ by Comparison ~~test~~

Method 2: use Integral test: $\frac{\ln n}{n} \geq 0$, $\frac{\ln n}{n}$ continuous,

$\left(\dfrac{\ln x}{x}\right)' = \dfrac{\frac{1}{x}x - \ln x}{x^2} = \dfrac{1-\ln x}{x^2} < 0$ for $x \geq 3$, so $\frac{\ln n}{n}$ is decr

$\int^\infty \frac{\ln x}{x}\,dx = \int^\infty u\,du = \frac{1}{2}u^2 \Big|^\infty$ ... Σ diverges

3. Do the following series converge absolutely, conditionally or do they diverge. Explain

3 a) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$, doesn't conv. abs. (see part c) of #2)

$\frac{\ln n}{n} \to 0$ (L'H), and $\frac{\ln n}{n}$ is decr. (see 2c))

so Σ converges cond. by alt. series test

3 b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$\sum \left|\frac{(-1)^n}{n^2}\right| = \sum \frac{1}{n^2}$ is a p=2 series, so converges,

hence $\sum \frac{(-1)^n}{n^2}$ conv. abs

3 c) $\sum_{n=1}^{\infty} (-1)^n n^2$

$(-1)^n n^2 \not\to 0$, so $\sum (-1)^n n^2$ diverges

5 4. a) Find a power series representation for $f(x) = \tan^{-1}(3x)$. (Hint: use geometric series formula to get going)

3 b) Find its radius of convergence

see attached

~~c)~~ What is its interval of convergence. (Don't forget to check the endpoints)

4 5. Find the Taylor series for $f(x) = 5x^3$ centered at 7 see attached

4 6. Evaluate, using series

$$\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$$

see attached

7. What are my first and last name?

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#4 a)

$$\tan^{-1}(3x) = \int \frac{3}{1+(3x)^2}\,dx$$

$|-9x^2|<1$

$$= 3\int \frac{1}{1-(-9x^2)}\,dx$$

$$= 3\int\left(\sum_{n=0}^{\infty}(-9x^2)^n\right)dx = 3\int\sum_{n=0}^{\infty}\left(9^n(-1)^n x^{2n}\right)dx$$

$$= 3\sum_{n=0}^{\infty}9^n(-1)^n\int x^{2n}\,dx = 3\sum_{n=0}^{\infty}(-1)^n 9^n \frac{x^{2n+1}}{2n+1}$$

$$= \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{3\cdot 9^n}{2n+1}x^{2n+1}}$$

b) $|-9x^2|<1 \iff |x^2|<\frac{1}{9} \iff |x|<\frac{1}{3}$

so $\boxed{R=\frac{1}{3}}$

c) check the end pts $x=-\frac{1}{3}$

$$\sum_{n=0}^{\infty}\frac{(-1)^n 3\cdot 9^n\left(-\frac{1}{3}\right)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n \cancel{3}\cdot\cancel{9^n}(-1)\cancel{\left(\frac{1}{3}\right)^{2n}}\cancel{\frac{1}{3}}}{2n+1}$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \text{ conv. by alt. s. t.}$$

$x=\frac{1}{3}$, $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\cancel{3\cdot 9^n}\cancel{\left(\frac{1}{3}\right)^{2n+1}}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ also conv. by alt. s. t.

Hence interval of convergence is

$$\boxed{\left[-\frac{1}{3},\frac{1}{3}\right]}$$

5. $f(x) = 5x^3, \quad a = 7$

$$f(a) = 5 \cdot 7^3$$

$$f'(a) = 15a^2 = 15 \cdot 7^2$$

$$f''(a) = 30a = 30 \cdot 7$$

$$f'''(a) = 30$$

$$f^{iv}(a) = 0$$

$$\text{T.S.} = \sum_{n=0}^{\infty} \frac{f^{(n)}(7)}{n!}(x-7)^n$$

$$= \boxed{5 \cdot 7^3 + \frac{15 \cdot 7^2}{1!}(x-7) + \frac{30 \cdot 7}{2!}(x-7)^2 + \frac{30}{3!}(x-7)^3}$$

6. $\lim\limits_{x \to 0} \dfrac{\sin x - x + \frac{1}{6}x^3}{x^5}$

$$= \lim_{x \to 0} \frac{\left(\cancel{x} - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) - \cancel{x} + \cancel{\frac{1}{6}x^3}}{x^5}$$

$$= \lim_{x \to 0} \left(\frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right)\frac{1}{x^5}$$

$$= \lim_{x \to 0} \frac{1}{5!} - \underbrace{x^2}_{0}\left(\frac{1}{7!} + \frac{x^2}{9!} - \frac{x^4}{11!} + \cdots\right)$$

$$= \frac{1}{5!} = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} = \boxed{\frac{1}{120}}$$