3. $0 < x < \pi$

$0 < y < \pi$

$u_y = 0$ for $y = 0, y = \pi$

$u = 0$, $x = 0$

$u(\pi, y) = \cos^2 y = \frac{1}{2}(1 + \cos 2y)$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

if $\lambda > 0$, $Y''(y) + \lambda Y(y) = 0$

has soln $Y(y) = A\cos\sqrt{\lambda}y + B\sin\sqrt{\lambda}y$

$0 = Y'(0) = B\sqrt{\lambda}\cos(\sqrt{\lambda}\cdot 0)$

$\Rightarrow B = 0$

$Y(y) = A\cos\sqrt{\lambda}y$

$0 = Y'(\pi) = \underbrace{-A\sqrt{\lambda}}_{\neq 0}\sin\sqrt{\lambda}\pi$

and $Y_n(y) = A_n\cos ny$

$\Rightarrow \sin\sqrt{\lambda}\pi = 0$

$\sqrt{\lambda}\pi = n\pi \Rightarrow \sqrt{\lambda} = n \Rightarrow \lambda = n^2$,

$n = 1, 2, 3, \ldots$

if $\lambda = 0$, $\quad Y''(y) + 0 \cdot Y(y) = 0$

$$Y''(y) = 0$$

$$Y(y) = A + By$$

$$Y'(y) = B$$

$$\left.\begin{array}{l} 0 = Y'(0) = B \Rightarrow B = 0 \\ 0 = Y'(\pi) = B \Rightarrow B = 0 \end{array}\right\} \text{same}$$

so $\lambda = 0$ is a legitimate eigenvalue & corresponds to $Y_0 = A$

λ cannot be negative or complex (not real) (see done before for Neumann BC)

$$\frac{X''(x)}{X(x)} = \lambda$$

$\underline{\lambda > 0:}$ $\quad X''(x) - \lambda X(x) = 0$

$\lambda = n^2$ (from Y-part)

$$X_n(x) = C_n \cosh nx + D_n \sinh nx$$

$$0 = X_n(0) = C_n \Rightarrow C_n = 0$$

$$X_n(x) = \sinh nx \quad , n = 1, 2, 3, \ldots$$

$\lambda = 0, \quad X''(x) = 0$

$$X(x) = C + \Delta x$$

$$0 = X(0) = C \Rightarrow C = 0$$

$$X_0(x) = \Delta_0 x$$

$$u(x,y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh nx \cos ny$$

$(\text{where } A_0 = A\Delta_0)$

$$\underbrace{u(\pi,y)}_{\frac{1}{2}(1+\cos 2y)} = A_0\pi + \sum_{n=1}^{\infty} A_n \sinh n\pi \cos ny$$

$$\frac{1}{2} + \frac{\cos 2y}{2} = A_0\pi + A_1 \sinh \pi \cos y + A_2 \sinh 2\pi \cos 2y + A_3 \sinh 3\pi \cos 3y + \cdots$$

by uniqueness of Fourier coefficients

$$\begin{cases} \frac{1}{2} = A_0\pi & \Rightarrow A_0 = \frac{1}{2\pi} \\ 0 = A_1 & \\ \frac{1}{2} = A_2 \sinh 2\pi & \Rightarrow A_2 = \frac{1}{2\sinh 2\pi} \\ 0 = A_n, & n \geq 3 \end{cases}$$

so $u(x,y) = \frac{1}{2\pi}x + \frac{\sinh 2x}{2\sinh 2\pi}\cos 2y$