MATHEMATICS 255

MIDTERM EXAM ANSWERS April 30, 2008

(1) [5 points each] Solve the following systems of equations:

(a) $\begin{bmatrix} 3x_1 + 2x_2 + 3x_3 = & 0 \\ x_1 \qquad\quad + \; x_3 = & -1 \end{bmatrix}$ `Answer:` $x_1 = -1 - t$, $x_2 = 3/2$, $x_2 = t$

(b) $\begin{bmatrix} -x \qquad - z = -5 \\ \quad - y + z = \;\; 3 \\ x - y - z = \;\; 2 \end{bmatrix}$ `Answer:` $x = 3$, $y = -1$, $z = 2$

(c) $\begin{bmatrix} \qquad b - c = -1 \\ a \qquad + c = \;\; 1 \\ 2a + b + c = \;\; 2 \end{bmatrix}$ `Answer:` *No solution*

(2) [5 points] Find the inverse of $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}$ `Answer:` $A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(3) [5 points] Find the rank of $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{bmatrix}$ `Answer:` 2

(4) [5 points] A system of 4 equations in 3 unknowns has no solutions. The coefficient matrix of this system has rank 2. Give a possible reduced form for the augmented matrix of this system. (Many answers are possible.)

`Answer:` *For example,* $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(5) [2 points each] Which of the following statements are true about any 4×4 matrices A and B and vector $b \in \mathbb{R}^4$: (Indicate either T=**always true** or F=**not always true**. No explanation needed)

(a) $AB = BA$ `Answer:` F
(b) If AB is invertible, then A and B are invertible. `Answer:` T
(c) If A and B are invertible, then $A + B$ is invertible. `Answer:` F
(d) If the rank of A is 4, then the system $Ax = b$ has a solution. `Answer:` T
(e) If the rank of A is 3, then the system $Ax = 0$ has infinitely many solutions. `Answer:` T
(f) If the rank of $[A|b]$ is 3 and the rank of A is 2, then the system $Ax = b$ has infinitely many solutions. `Answer:` F
(g) If the columns of A are independent, then there is a unique matrix C such that $AC = B$. `Answer:` T

(h) If the columns of A span $\mathbb{R}^4$, then A is invertible. `Answer:` T

(i) If the rank of A is zero, then $A = 0$. `Answer:` T

(j) If A is the augmented matrix of a system of equations and A is invertible, then the system of equations has a unique solution. `Answer:` F

(6) [5 points each] A matrix A and its reduced form R are as follows:

$$A = \begin{bmatrix} \overset{v_1}{1} & \overset{v_2}{2} & \overset{v_3}{1} & \overset{v_4}{3} & \overset{v_5}{1} & \overset{v_6}{12} \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 1 & 0 & 2 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Let $v_1, v_2, \ldots, v_6$ be the column vectors of A.

(a) Does the set $\{v_1, v_2, v_3, v_4\}$ span $\mathbb{R}^3$? Explain. `Answer:` *No. the reduced form of the matrix with these vectors as columns does not have a leading one in the bottom column.*

(b) Is the set $\{v_1, v_2, v_3, v_4\}$ independent? If not, write a nontrivial linear combination of these vectors that is zero. `Answer:` *No. Any set of* 4 *vectors in* $\mathbb{R}^3$ *is dependent. For example,* $2v_1 - v_2 = 0$.

(c) Express v_6 as a linear combination of v_1, v_3 and v_5. `Answer:` $v_6 = 2v_1 + 3v_3 + 7v_5$

(d) Is v_6 in $\text{Span}\{v_4, v_5\}$? Explain. `Answer:` *No. The augmented matrix for the equation* $c_4v_4 + c_5v_5 = v_6$ *reduces to the identity matrix. Since the last column contains a leading one, there is no solution.*

$$\begin{bmatrix} 3 & 1 & 12 \\ 2 & 0 & 3 \\ 1 & 0 & 2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \\ 0 & 1 & 7 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

OR Any vector of the form $c_4v_4 + c_5v_5$ *has the property that its middle component is twice the bottom component. Since* v_6 *doesn't have this property, it can't be of the form* $c_4v_4 + c_5v_5$.