Math 255 May 19, 2008

QUIZ SOLUTIONS

QUIZ 1

(1) Indicate whether the following matrices are in row echelon form, row-reduced echelon form, or neither.

(a) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

`Answer:` *(a) echelon, (b) neither, (c) reduced*

(2) Solve the following systems of equations:

(a) $\left[\begin{array}{rcrcrcr} & - & y & + & z & = & 3 \\ x & - & y & - & z & = & 0 \\ -x & & & - & z & = & -3 \end{array}\right]$ `Answer:` $x = 1$, $y = -1$, $z = 2$

(b) $\left[\begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 1 \\ & & x_2 & - & x_3 & = & -1 \\ 2x_1 & + & x_2 & + & x_3 & = & 2 \end{array}\right]$ `Answer:` *No solution*

QUIZ 2

A matrix A and its reduced form R are as follows:

$$A = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \left[\begin{array}{rrrrrr} 1 & -1 & 2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 4 & 1 & 2 \\ 7 & -2 & 4 & 8 & 1 & 6 \end{array}\right] \end{array} \qquad R = \left[\begin{array}{rrrrrr} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

Let $v_1, v_2, v_3, v_4, v_5, v_6$ be the column vectors of A.

(1) What is $\mathrm{Span}\{v_1, v_2, v_3, v_4, v_5, v_6\}$? Explain. `Answer:` *$\mathbb{R}^3$, since every row of R has a leading one.*

(2) Find a smaller subset of these vectors whose span is $\mathrm{Span}\{v_1, v_2, v_3, v_4, v_5, v_6\}$. `Answer:` *For example, $\{v_1, v_2, v_4\}$. This set spans $\mathbb{R}^3$ too.*

(3) Is v_5 a linear combination of the other vectors? `Answer:` *Yes. The reduced form of the augmented matrix for the equation $c_1v_1 + c_2v_2 + c_4v_4 = v_5$ is*

$$\left[\begin{array}{rrrr} 1 & -1 & 3 & 1 \\ 3 & -1 & 4 & 1 \\ 7 & -2 & 8 & 1 \end{array}\right] \mapsto \left[\begin{array}{rrrr} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Hence $c_1 = -1$, $c_2 = 4$, $c_4 = 2$ and $-v_1 + 4v_2 + 2v_4 = v_5$.

(4) Is v_6 in $\mathrm{Span}\{v_1, v_3\}$? If so, write v_6 as a linear combination of v_1 and v_3. `Answer:` *No. The reduced form of the augmented matrix for the equation $c_1v_1 + c_3v_3 = v_6$ has a leading one in the last column:*

$$\left[\begin{array}{rrr} 1 & 2 & -1 \\ 3 & 2 & 2 \\ 7 & 4 & 6 \end{array}\right] \mapsto \left[\begin{array}{rrr} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array}\right] \mapsto \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

(5) Is v_6 in $\mathrm{Span}\{v_1, v_4\}$? If so, write v_6 as a linear combination of v_1 and v_4. `Answer:` *Yes. The reduced form of the augmented matrix for the equation $c_1v_1 + c_4v_4 = v_6$ has no leading one in the last column. From the reduced form we get $c_1 = 2$ and $c_4 = -1$, so $v_6 = 2v_1 - v_4$.*

$$\left[\begin{array}{rrr} 1 & 3 & -1 \\ 3 & 4 & 2 \\ 7 & 8 & 6 \end{array}\right] \mapsto \left[\begin{array}{rrr} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{array}\right] \mapsto \left[\begin{array}{rrr} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

QUIZ 3

A matrix A and its reduced form R are as follows:

$$A = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 1 & -1 & 2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 4 & 1 & 2 \\ 7 & -2 & 4 & 8 & 1 & 6 \end{bmatrix} \end{array} \qquad R = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$$

Let $v_1, v_2, v_3, v_4, v_5, v_6$ be the column vectors of A.

(1) Is $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ independent? Explain. `Answer:` *No, since not every column of R has a leading one. OR* <u>*Any*</u> *set of more than 3 vectors of $\mathbb{R}^3$ is dependent.*

(2) Find two vectors from $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ that are dependent. `Answer:` $\{v_2, v_3\}$ *since each is a multiple of the other (or since $2v_1 + v_2 = 0$).*

(3) Is $\{v_1, v_3, v_6\}$ independent? If not, write a nontrivial linear combination of these vectors that is zero. `Answer:` *Yes. The reduced form of the matrix whose columns are v_1, v_2 and v_6 has a leading one in each column.*

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 2 \\ 7 & 4 & 6 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(4) Is $\{v_1, v_4, v_6\}$ independent? If not, write a nontrivial linear combination of these vectors that is zero. `Answer:` *No. The reduced form of the matrix whose columns are v_1, v_4 and v_6 has no leading in the last column.*

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 2 \\ 7 & 8 & 6 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

From the reduced matrix we get the equation $2v_1 - v_4 - v_6 = 0$.

QUIZ 4

(1) Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ `Answer:` $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

(2) For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, find its reduced form R and a matrix P such that $R = PA$.

`Answer:` $R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

(3) With A and R as in (2), find a matrix Q such that $A = QR$.

`Answer:` $Q = P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

QUIZ 5

(1) What is the standard matrix for $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y - x \end{bmatrix}$?

`Answer:` $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

(2) Suppose T is linear, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$. Find $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

`Answer:` $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(3) Find the standard matrix for T from Question (2).

`Answer:` $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$

QUIZ 6

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y + z \\ x - y \end{bmatrix} \qquad U\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} u + v \\ u - v \\ 0 \end{bmatrix}.$$

(1) Is T one-to-one? Explain. `Answer:` *No. The standard matrix for T is* $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$. *Since this matrix has more columns than rows, it is not possible that each column of the reduced form have a leading one. Hence T is not one-to-one.*

(2) Is U onto? Explain. `Answer:` *No. For example, the vector* $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ *is not U of any vector in $\mathbb{R}^2$.*

(3) Is $T \circ U$ invertible? Explain. `Answer:` *Yes. The standard matrix for $T \circ U$ is* $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, *an obviously invertible matrix. Hence T is invertible.*

QUIZ 7

(1) For which c is the following matrix $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & c \\ 0 & c & -15 \end{bmatrix}$ NOT invertible? `Answer:` $c = 3, -5$

(2) Calculate the determinant of $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix}$. `Answer:` 18