

Antipodal Labelings for Cycles

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Abstract

Let G be a graph with diameter d . An antipodal labeling of G is a function f that assigns to each vertex a non-negative integer (label) such that for any two vertices u and v , it is satisfied that $|f(u) - f(v)| \geq d - d(u, v)$, where $d(u, v)$ is the distance between u and v . The span of an antipodal labeling f is $\max\{f(u) - f(v) : u, v \in V(G)\}$. The antipodal number for G , denoted by $\text{an}(G)$, is the minimum span of an antipodal labeling for G . Let C_n denote the cycle on n vertices. Chartrand, Erwin and Zhang [4] determined the value of $\text{an}(C_n)$ for $n \equiv 2 \pmod{4}$. In this article we determine the value of $\text{an}(C_n)$ for $n \equiv 1 \pmod{4}$, confirming a conjecture of Chartrand et al. Moreover, we settle the case $n \equiv 3 \pmod{4}$; and improve the known lower bound and give an upper bound for the case $n \equiv 0 \pmod{4}$.

1 Introduction

Radio k -labeling was motivated by the frequency assignment problem (cf. [7]). Let k be a positive integer. A *radio k -labeling* (or *k -labeling* for short) for a graph G is a function, $f : V(G) \rightarrow \{0, 1, 2, \dots\}$, such that the following is satisfied for any vertices u and v :

$$|f(u) - f(v)| \geq k + 1 - d(u, v).$$

where $d(u, v)$ denotes the distance between u and v . The *span* of such a function f , denoted by $\text{sp}(f)$, is defined as $\text{sp}(f) = \max\{f(u) - f(v) : u, v \in V(G)\}$. The minimum span over all k -labelings of a graph G is called the Φ_k -number and denoted by $\Phi_k(G)$.

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For the special case that $k = 1$, the 1-labeling is indeed the conventional vertex coloring and we have $\Phi_1(G) = \chi(G) - 1$, where $\chi(G)$ is the chromatic number of G . Another special case is when $k = 2$, the 2-labeling is the same as the *distance two labeling* (or $L(2, 1)$ -labeling) which has been studied extensively in the past years (cf. [1, 2, 3, 9, 10, 11, 12, 14]). The Φ_2 -number is known as the λ -number of G .

The radio k -labeling for large values of k has also been investigated by several authors. Let G be a connected graph. The maximum distance among all pairs of vertices in G is the *diameter* of G , denoted by $\text{diam}(G)$. The *radio labeling* (or *multi-level distance labeling*) is a radio k -labeling when $k = \text{diam}(G)$. The $\Phi_{\text{diam}(G)}$ -number of G is called the *radio number* of G , denoted by $\text{rn}(G)$. The radio number for different families of graphs has been investigated in [6, 8, 15, 16, 17, 18, 19]. For instance, the radio number for paths and cycles has been studied in [6, 8, 19] and was recently settled in [18].

The case for $k = \text{diam}(G) - 1$, a k -labeling is called an *antipodal labeling*. That is, an *antipodal labeling* (or *radio antipodal coloring*) for G is a function, $f : V(G) \rightarrow \{0, 1, 2, \dots\}$, such that the following is satisfied for any two vertices u and v :

$$|f(u) - f(v)| \geq \text{diam}(G) - d(u, v).$$

The *antipodal number* for G , denoted by $\text{an}(G)$, is the minimum span of an antipodal labeling admitted by G . Notice that a radio labeling is a one-to-one function; while in an antipodal labeling, two vertices of distance $\text{diam}(G)$ apart may receive the same label (this is where the name ‘‘antipodal’’ came from).

The antipodal labeling for graphs was first studied by Chartrand et al. [4, 5], in which, among other results, general bounds of $\text{an}(G)$ were obtained. Khennoufa and Togni [13] determined the exact value of $\text{an}(P_n)$ for paths P_n . The antipodal labeling for cycles C_n was studied in [4], in which lower bounds for $\text{an}(C_n)$ were shown. In addition, the bound for the case $n \equiv 2 \pmod{4}$ was proved to be the exact value of $\text{an}(C_n)$, and the bound for the case $n \equiv 1 \pmod{4}$ was conjectured to be the exact value as well [4].

In this article, we confirm the conjecture mentioned above. Moreover, we determine the value of $\text{an}(C_n)$ for the case $n \equiv 3 \pmod{4}$. For the case $n \equiv 0 \pmod{4}$, we improve the known lower bound [4] and give an upper bound. It is conjectured that the upper bound is the exact value.

2 Lower Bounds

In this section, we establish lower bounds for $\text{an}(C_n)$. These bounds were proved by Chartrand et al [4]. We present here a different proof which includes techniques that will be used in later sections.

In an antipodal labeling, the number assigned to a vertex is called a *label*. Notice that as we are seeking for the minimum span of an antipodal labeling, without loss of generality we assume that the label 0 is used by any antipodal labeling. Consequently, the span of f is the maximum label used.

In the following we introduce notations to be used throughout this article. Denote $V(C_n) = \{v_0, v_1, \dots, v_{n-1}\}$, where $v_i v_{i+1} \in E(C_n)$ for $0 \leq i \leq n-2$, and $v_{n-1} v_0 \in E(C_n)$. The diameter of C_n is denoted by d , where $d = \lfloor n/2 \rfloor$. Every antipodal labeling f for C_n gives an ordering of the vertices according to the labels assigned. Denote the ordering by $(x_0, x_1, \dots, x_{n-1})$, where $\{x_0, x_1, \dots, x_{n-1}\} = V(C_n)$ and

$$0 = f(x_0) \leq f(x_1) \leq f(x_2) \leq \dots \leq f(x_{n-1}).$$

Note, the span of f is $f(x_{n-1})$.

For $i = 0, 1, \dots, n-2$, we define the *distance gap* and *label gap*, respectively, by:

$$d_i = d(x_i, x_{i+1}), \quad f_i = f(x_{i+1}) - f(x_i).$$

By definition, it holds that $f_i \geq d - d_i$.

Proposition 1 *For any three vertices u, v and w on a cycle C_n ,*

$$d(u, v) + d(v, w) + d(u, w) \leq n.$$

Proof. Without loss of generality, assume $d(u, v), d(v, w) \leq d(u, w)$. If all the three vertices lie on one half of the cycle, then $d(u, v) + d(v, w) + d(u, w) = 2d(u, w) \leq n$. Otherwise, we have $d(u, v) + d(v, w) + d(u, w) = n$. \square

Lemma 2 *Let f be an antipodal labeling for C_n , $n \geq 3$, with labels $f(x_0) \leq f(x_1) \leq \dots \leq f(x_{n-1})$. Let $n = 4k + r$ for some $0 \leq r \leq 3$. Then for any $0 \leq i \leq n-3$,*

$$f(x_{i+2}) - f(x_i) = f_i + f_{i+1} \geq \begin{cases} k, & \text{if } r = 0, 1, 3; \\ k + 1, & \text{if } r = 2. \end{cases}$$

Proof. By definition, we have $f(x_{i+1}) - f(x_i) \geq d - d(x_{i+1}, x_i)$, $f(x_{i+2}) - f(x_{i+1}) \geq d - d(x_{i+2}, x_{i+1})$, and $f(x_{i+2}) - f(x_i) \geq d - d(x_{i+2}, x_i)$. Summing up these three in-equalities and by Proposition 1, we get

$$2(f(x_{i+2}) - f(x_i)) \geq 3d - (d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + d(x_i, x_{i+2})) \geq 3d - n.$$

Therefore, $f_i + f_{i+1} = f(x_{i+2}) - f(x_i) \geq \lceil (3d - n)/2 \rceil$. The results then follow by immediate calculations for different values of n . \square

Corollary 3 [4] *Let $n = 4k + r$ for some $n \geq 3$ and $0 \leq r \leq 3$. Then*

$$\text{an}(C_n) \geq \begin{cases} k(2k - 1), & \text{if } r = 0; \\ 2k^2, & \text{if } r = 1; \\ 2k(k + 1), & \text{if } r = 2; \\ k(2k + 1), & \text{if } r = 3. \end{cases}$$

Proof. Let f be an antipodal labeling for C_n . The span of f is

$$f(x_{n-1}) = f_0 + f_1 + \cdots + f_{n-2}.$$

By Lemma 2, the results follow by pairing up the terms in the above summation and leaving the last term f_{n-2} (if n is even) which is at least 0. \square

In [4], it was proved that the equality in Corollary 3 holds for the case $n \equiv 2 \pmod{4}$, and conjectured that the equality also holds for the case $n \equiv 1 \pmod{4}$. This conjecture is confirmed in the next section.

3 $n = 4k + 1$

Let f be an antipodal labeling for a cycle C_n with $0 = f(x_0) \leq f(x_1) \leq \cdots \leq f(x_{n-1})$. In the rest of this article, we denote the permutation π on $\{0, 1, 2, \dots, n-1\}$ generated from f by

$$\pi(i) = j \iff x_i = v_j.$$

For an integer x and a positive integer y , we denote “ $x \bmod y$ ” as a binary operation which outputs an integer z with $z \equiv x \pmod{y}$ and $0 \leq z \leq y - 1$.

In this section, we prove the following result.

Theorem 4 *If $n = 4k + 1$ for some integer $k \geq 1$, then $\text{an}(C_n) = 2k^2$.*

Proof. By Corollary 3, it suffices to find an antipodal labeling with span $2k^2$. Two cases are considered. Recall $d = \text{diam}(C_{4k+1}) = 2k$.

Case 1. k is odd First, we label the $2k + 1$ vertices x_0, x_2, \dots, x_{4k} by

$$\pi(2i) = ki \bmod n, \text{ and } f(x_{2i}) = ki, \text{ for } i = 0, 1, 2, \dots, 2k.$$

For instance, $\pi(2) = k$ (i.e., $x_2 = v_k$) and $f(x_2) = k$; and $\pi(4k) = 2k - \frac{k-1}{2}$ and $f(x_{4k}) = 2k^2$.

Secondly, we label the remaining vertices $x_1, x_3, \dots, x_{4k-1}$ by $\pi(1) = \pi(4k) + k = 3k - \frac{k-1}{2}$; and $\pi(2i + 1) = (\pi(2i - 1) + k) \bmod n$, for $i = 1, 2, \dots, 2k - 1$, with labels $f(x_{2i+1}) = (k - 1)/2 + ki$ for $i = 0, 1, \dots, 2k - 1$. See Figure 1 for an example. In Figure 1 (and all other figures), the number inside the circle for each vertex is the label assigned to that vertex.

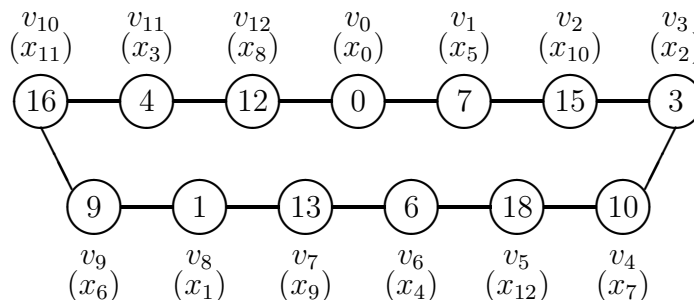


Figure 1: An antipodal labeling for C_{13} with minimum span $\text{an}(C_{13}) = 18$.

To see that π is a permutation of $\{0, 1, \dots, n - 1\}$, we observe that $\pi(0), \pi(2), \dots, \pi(4k), \pi(1), \pi(3), \dots, \pi(4k - 1)$ is a list of vertices winding around C_n by jumping k vertices between any two consecutive terms. Since $\text{gcd}(n, k) = 1$, so π is a permutation of $\{0, 1, \dots, n - 1\}$. In addition, one can easily check that for every i the following hold:

$$\begin{aligned} f(x_{i+1}) - f(x_i) &\geq d - d(x_{i+1}, x_i), \\ f(x_{i+2}) - f(x_i) &= k = 2k - k = d - d(x_{i+2}, x_i), \\ f(x_{i+s}) - f(x_i) &\geq 2k \geq d - d(x_{i+s}, x_i), \text{ for } s \geq 4. \end{aligned}$$

Hence, to show that f is an antipodal labeling, it suffices to verify $f(x_{i+3}) - f(x_i) \geq 2k - d(x_{i+3}, x_i)$. This is true since $d(x_{i+3}, x_i) = (k + 1)/2$, and $f(x_{i+3}) - f(x_i) \in \{(3k - 1)/2, (3k + 1)/2\}$.

Case 2. k is even Similar to Case 1, we first label the $2k + 1$ vertices x_0, x_2, \dots, x_{4k} , by $\pi(2i) = ki \bmod n$, for $i = 0, 1, \dots, 2k$, using labels $f(x_{2i}) = ki$. Note that since $2k^2 \equiv n - \frac{k}{2} \pmod{n}$, we have $x_{4k} = v_{n-(k/2)}$.

Secondly, we label the remaining vertices by $\pi(1) = 2k + 1$, $f(x_1) = 0$, and

$$\pi(2i + 1) = \begin{cases} (\pi(2i - 1) + k) \pmod{n}, & \text{if } i \text{ is odd;} \\ (\pi(2i - 1) + k + 1) \pmod{n}, & \text{if } i \text{ is even,} \end{cases}$$

with labels

$$f(x_{2i+1}) = \begin{cases} f(x_{2i-1}) + k, & \text{if } i \text{ is odd;} \\ f(x_{2i-1}) + k + 1, & \text{if } i \text{ is even.} \end{cases}$$

See Figure 2 for an example.

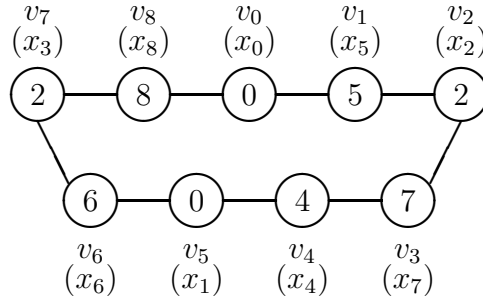


Figure 2: An antipodal labeling for C_9 with minimum span $\text{an}(C_9) = 8$.

By some calculation, one gets $\pi(1) = 2k + 1 \equiv -2k \pmod{n}$, and for $1 \leq i \leq 2k - 1$,

$$\pi(2i + 1) \equiv \begin{cases} -ik \pmod{n}, & \text{if } i \text{ is odd;} \\ -(i + 2)k \pmod{n}, & \text{if } i \text{ is even.} \end{cases}$$

Since $\pi(2i) = ki \bmod n$, for $0 \leq i \leq 2k$, we conclude

$$\{\pi(i) : 0 \leq i \leq 4k\} = \{jk \bmod n : -2k \leq j \leq 2k\}.$$

Since $\gcd(n, k) = 1$, π is a permutation. Similar to Case 1, it is straightforward to check that f is an antipodal labeling, and we shall leave the details to the reader.

4 $n = 4k + 3$

As it turned out (Theorem 5), the exact value of $\text{an}(C_{4k+3})$ is greater than the lower bound established in Corollary 3.

Theorem 5 *For every integer $k \geq 0$, $\text{an}(C_{4k+3}) = 2k^2 + 2k$.*

Note, when $k = 0$ in Theorem 5, it is trivial that $\text{an}(C_3) = 0$. The following lemma will be used to prove Theorem 5 for $k \geq 1$.

Lemma 6 *Let f be an antipodal labeling for C_n where $n = 4k + 3$, $k \geq 1$. If $f_i + f_{i+1} = k$ for some $0 \leq i \leq n - 3$, then the following hold:*

$$(1) \quad d(x_i, x_{i+2}) = k + 1,$$

$$(2) \quad f_i = t, \quad d_{i+1} = k + t + 1, \quad \text{and} \quad d_i = 2k - t + 1, \quad \text{for some } t \in \{0, 1, \dots, k\}.$$

Proof. Recall $d = \text{diam}(C_{4k+3}) = 2k + 1$. Assume $f_i + f_{i+1} = k$ for some i . By definition,

$$d(x_i, x_{i+2}) \geq d - (f(x_{i+2}) - f(x_i)) = d - (f_{i+1} + f_i) = (2k + 1) - k = k + 1.$$

On the other hand, by Proposition 1 and definition, we have

$$\begin{aligned} d(x_i, x_{i+2}) &\leq (4k + 3) - (d_i + d_{i+1}) \\ &\leq (4k + 3) - (d - f_i + d - f_{i+1}) \\ &= (4k + 3) - (4k + 2 - k) \\ &= k + 1. \end{aligned}$$

This verifies (1).

Let $f_i = t$ for some $t \in \{0, 1, \dots, k\}$. By (1), the second equality in the above holds, which implies that $d_i = d - f_i$ and $d_{i+1} = d - f_{i+1}$. Therefore, (2) follows as $d = 2k + 1$. \square

Lemma 7 *Let f be an antipodal labeling for C_n where $n = 4k + 3$ for some integer $k \geq 1$. Then for any $0 \leq i \leq n - 5$,*

$$f_i + f_{i+1} + f_{i+2} + f_{i+3} \geq 2k + 1.$$

Proof. Assume to the contrary that for some i , $f_i + f_{i+1} + f_{i+2} + f_{i+3} \leq 2k$. By Lemma 2, $f_i + f_{i+1} = f_{i+2} + f_{i+3} = k$. By symmetry and by Lemma 6 (1), without loss of generality, assume $x_i = v_0$, $x_{i+2} = v_{k+1}$ and $x_{i+4} = v_{2(k+1)}$. By Lemma 6 (2), $f_i = t$ for some $0 \leq t \leq k$ and $d_i = 2k - t + 1$. Note, $x_{i+1} \neq v_{2k-t+1}$, for otherwise it would be $d_{i+1} = d(x_{i+1}, x_{i+2}) = k - t$, a contradiction. Hence, we conclude $x_{i+1} = v_{n-(2k-t+1)} = v_{2k+t+2}$. This implies $d(x_{i+4}, x_i) = t$. Because f is an antipodal labeling, we have

$$2k - t = f_{i+1} + f_{i+2} + f_{i+3} = f(x_{i+4}) - f(x_{i+1}) \geq 2k + 1 - d(x_{i+4}, x_{i+1}) = 2k + 1 - t,$$

a contradiction. \square

Theorem 8 For every integer $k \geq 1$, $\text{an}(C_{4k+3}) \geq 2k^2 + 2k$.

Proof. By Lemmas 2 and 7, the span of an antipodal labeling f for C_{4k+3} has

$$\begin{aligned} f(x_{4k+2}) - f(x_0) &= f_0 + f_1 + \cdots + f_{4k+1} \\ &= \sum_{i=0}^{k-1} (f_{4i} + f_{4i+1} + f_{4i+2} + f_{4i+3}) + f_{4k} + f_{4k+1} \\ &\geq k(2k + 1) + k = 2k^2 + 2k. \end{aligned}$$

\square

Proof of Theorem 5. For $k = 0$, $\text{an}(C_3) = 0$ is trivial as mentioned earlier. For $k \geq 1$, it remains to find an antipodal labeling for C_{4k+3} with span equal to the desired number. First, we label the vertices $x_0, x_2, \dots, x_{4k+2}$, by $\pi(0) = 0$ and $f(x_0) = 0$; and for $1 \leq i \leq 2k + 1$,

$$\pi(2i) = \begin{cases} (\pi(2i - 2) + k + 1) \pmod n, & \text{if } i \text{ is odd;} \\ (\pi(2i - 2) + k) \pmod n, & \text{if } i \text{ is even,} \end{cases}$$

$$f(x_{2i}) = \begin{cases} f(x_{2i-2}) + k, & \text{if } i \text{ is odd;} \\ f(x_{2i-2}) + k + 1, & \text{if } i \text{ is even.} \end{cases}$$

Secondly, we label the remaining vertices by $\pi(1) = 2k + 2$ and $f(x_1) = 0$; and for $1 \leq i \leq 2k$,

$$\pi(2i + 1) = (\pi(2i - 1) + k + 1) \pmod n, \text{ and } f(x_{2i+1}) = i(k + 1).$$

See Figure 3 for an example.

By some calculation, one gets

$$\pi(2i + 1) \equiv (i + 2)(k + 1) \pmod n, \text{ for } 0 \leq i \leq 2k, \text{ and}$$

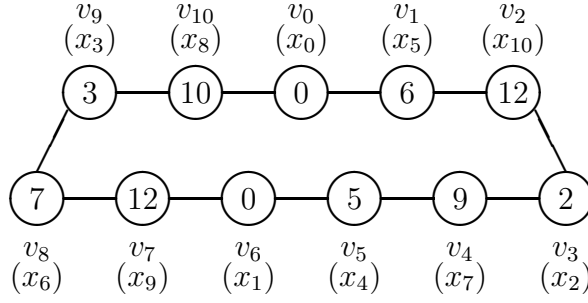


Figure 3: An antipodal labeling for C_{11} with minimum span $\text{an}(C_{11}) = 12$.

$$\pi(2i) \equiv \begin{cases} -(i-2)(k+1) \pmod{n}, & \text{if } i \text{ is odd;} \\ -i(k+1) \pmod{n}, & \text{if } i \text{ is even,} \end{cases}$$

for $0 \leq i \leq 2k+1$. Hence, we conclude

$$\{\pi(i) : 0 \leq i \leq 4k+2\} = \{j(k+1) \pmod{n} : -2k \leq j \leq 2k+2\}.$$

Because $\gcd(n, k+1) = 1$, π is a permutation. Similar to the proof of Theorem 4, it is straightforward to show that f is an antipodal labeling, and we shall leave the details to the reader. This completes the proof of Theorem 5. \square

5 $n = 4k$

Note, it is trivial that $\text{an}(C_4) = 1$. For cycles with $n = 4k$ nodes, $k \geq 2$, we improve the lower bound in Corollary 3 and give an upper bound.

Theorem 9 *For every integer $k \geq 2$,*

$$2k^2 - \lfloor k/2 \rfloor \leq \text{an}(C_{4k}) \leq 2k^2 - 1.$$

The following lemma will be used to prove the lower bound for $\text{an}(C_{4k})$ in Theorem 9. Recall that $d = \text{diam}(C_{4k}) = 2k$.

Lemma 10 *Let f be an antipodal labeling of C_{4k} , for some integer $k \geq 2$. If $f_i + f_{i+1} = k$ for some $0 \leq i \leq n-3$, then $d(x_i, x_{i+2}) = k$.*

Proof. Assume $f_i + f_{i+1} = k$ for some $0 \leq i \leq n - 3$. Then $d(x_i, x_{i+2}) \geq d - (f_i + f_{i+1}) = k$. On the other hand, by Proposition 1 and definition,

$$\begin{aligned} d(x_i, x_{i+2}) &\leq n - (d_i + d_{i+1}) \\ &\leq 4k - (d - f_i + d - f_{i+1}) \\ &= k. \end{aligned}$$

□

Lemma 11 *Let f be an antipodal labeling of C_{4k} , $k \geq 2$. Then for any $0 \leq i \leq n - 9$,*

$$\sum_{j=0}^7 f_{i+j} \geq 4k + 1.$$

Proof. Assume to the contrary that for some $0 \leq i \leq n - 9$, $\sum_{j=0}^7 f_{i+j} \leq 4k$. By Lemma 2, $f_i + f_{i+1} = f_{i+2} + f_{i+3} = f_{i+4} + f_{i+5} = f_{i+6} + f_{i+7} = k$. By Lemma 10, $d(x_i, x_{i+2}) = d(x_{i+2}, x_{i+4}) = d(x_{i+4}, x_{i+6}) = d(x_{i+6}, x_{i+8}) = k$. Since $n = 4k$, it is impossible that all these four equations hold. So the result follows. □

Lemma 12 *Let f be an antipodal labeling of C_{4k} , $k \geq 2$. All the following hold.*

- (1) *If $f_i + f_{i+1} = k$ for some $0 \leq i \leq n - 4$, then $f_{i+2} \geq f_i$.*
- (2) *If $f_i + f_{i+1} = k$ for some $1 \leq i \leq n - 3$, then $f_{i-1} \geq f_{i+1}$.*
- (3) *If $f_i + f_{i+1} + f_{i+2} + f_{i+3} = 2k$ for some $0 \leq i \leq n - 6$, then $f_{i+4} \geq f_i \geq 1$.*
- (4) *If $\sum_{j=0}^7 f_{i+j} = 4k + 1$ for some $0 \leq i \leq n - 10$, then $f_{i+8} \geq f_i$.*
- (5) *If $\sum_{j=0}^7 f_{i+j} = 4k + 1$ for some $0 \leq i \leq n - 10$, then $f_{i+8} \geq 1$.*
- (6) *For any $0 \leq i \leq n - 6$, $\sum_{j=0}^4 f_{i+j} \geq 2k + 1$.*
- (7) *For any $0 \leq i \leq n - 10$, $\sum_{j=0}^8 f_{i+j} \geq 4k + 2$.*

Proof. To prove (1), assume $f_i + f_{i+1} = k$ for some $0 \leq i \leq n - 4$. By Lemma 2, $f_{i+2} + f_{i+1} \geq k = f_{i+1} + f_i$, hence $f_{i+2} \geq f_i$. (2) follows by a similar argument.

To prove (3), assume $f_i + f_{i+1} + f_{i+2} + f_{i+3} = 2k$ for some $0 \leq i \leq n - 6$. Then by Lemma 2, $f_i + f_{i+1} = f_{i+2} + f_{i+3} = k$. By Lemma 10, $d(x_i, x_{i+2}) = d(x_{i+2}, x_{i+4}) = k$,

so $d(x_i, x_{i+4}) = 2k$. This implies that $d_i < 2k$, as $n = 4k$. By definition of antipodal labeling, $f_i \geq 1$. Hence, by (1), we have $f_{i+4} \geq f_{i+2} \geq f_i \geq 1$.

To prove (4), assume $\sum_{j=0}^7 f_{i+j} = 4k + 1$ for some $0 \leq i \leq n - 10$. By Lemma 11, $\sum_{j=1}^8 f_{i+j} \geq 4k + 1 = \sum_{j=0}^7 f_{i+j}$, hence $f_{i+8} \geq f_i$.

To prove (5), assume $\sum_{j=0}^7 f_{i+j} = 4k + 1$ for some $0 \leq i \leq n - 10$. By Lemma 2, we have $f_i + f_{i+1} = f_{i+2} + f_{i+3} = k$ or $f_{i+4} + f_{i+5} = f_{i+6} + f_{i+7} = k$. For the former case, the result follows by (4) and (3); for the latter case, the results follows by (3).

(6) follows by (3) and Lemma 2; and (7) follows by (5) and Lemma 11. \square

Corollary 13 For any integer $k \geq 2$, $\text{an}(C_{4k}) \geq 2k^2 - \lfloor k/2 \rfloor$.

Proof. For $k = 2$, by Lemma 2 and Lemma 12 (6), the span of an antipodal labeling f for C_8 has $f(x_7) = (f_0 + f_1 + \cdots + f_4) + (f_5 + f_6) \geq 5 + 2 = 2k^2 - \lfloor k/2 \rfloor$.

For $k \geq 3$, by Lemmas 2, 11 and 12 (7), the span of an antipodal labeling f for C_{4k} has

$$\begin{aligned} f(x_{4k-1}) &= \sum_{i=0}^8 f_i + \sum_{i=1}^{\lfloor (4k-9)/8 \rfloor} \left(\sum_{j=1}^8 (f_{8i+j}) \right) + f_{8\lfloor (4k-1)/8 \rfloor + 1} + f_{8\lfloor (4k-1)/8 \rfloor + 2} + \cdots + f_{4k-2} \\ &\geq \begin{cases} (4k + 2) + \lfloor 2k^2 - (11/2)k - 3/2 \rfloor + k, & \text{if } k \text{ is odd} \\ (4k + 2) + \lfloor 2k^2 - (15/2)k - 2 \rfloor + 3k, & \text{if } k \text{ is even} \end{cases} \\ &= 2k^2 - \lfloor k/2 \rfloor. \end{aligned}$$

\square

Proof of Theorem 9. It remains to find an antipodal labeling for C_{4k} with span $2k^2 - 1$. First, we label the vertices $x_0, x_2, \cdots, x_{4k-2}$, by $\pi(0) = 0$ and $f(x_0) = 0$; and for $1 \leq i \leq 2k - 1$,

$$\pi(2i) = \begin{cases} (\pi(2i - 2) + k) \pmod n, & \text{if } i \text{ is odd;} \\ (\pi(2i - 2) + k + 1) \pmod n, & \text{if } i \text{ is even,} \end{cases}$$

$$f(x_{2i}) = \begin{cases} f(x_{2i-2}) + k, & \text{if } i \text{ is odd;} \\ f(x_{2i-2}) + k + 1, & \text{if } i \text{ is even.} \end{cases}$$

Secondly, we label the remaining vertices by: For $0 \leq i \leq 2k - 1$,

$$\pi(2i + 1) = (\pi(2i) + 2k) \pmod n, \text{ and } f(x_{2i+1}) = f(x_{2i}).$$

See Figure 4 for an example.

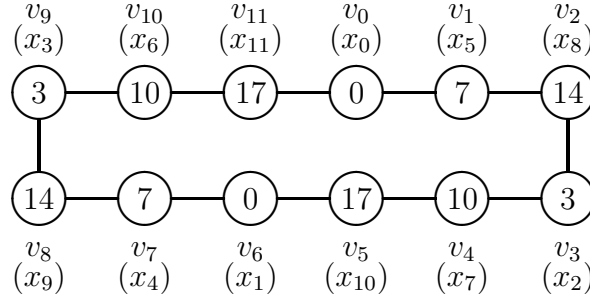


Figure 4: An antipodal labeling for C_{12} with minimum span $\text{an}(C_{12}) = 17$.

By some calculation, one gets the following for $0 \leq i \leq k - 1$:

$$\begin{aligned}
\pi(4i) &\equiv i(2k + 1) \pmod{n}. \\
\pi(4i + 1) &\equiv (i + 2k)(2k + 1) \pmod{n}. \\
\pi(4i + 2) &\equiv \begin{cases} (i + 3k)(2k + 1) \pmod{n}, & \text{if } k \text{ is odd;} \\ (i + k)(2k + 1) \pmod{n}, & \text{if } k \text{ is even.} \end{cases} \\
\pi(4i + 3) &\equiv \begin{cases} (i + k)(2k + 1) \pmod{n}, & \text{if } k \text{ is odd;} \\ (i + 3k)(2k + 1) \pmod{n}, & \text{if } k \text{ is even.} \end{cases}
\end{aligned}$$

Therefore, we conclude

$$\{\pi(i) : 0 \leq i \leq 4k - 1\} = \{j(2k + 1) \pmod{n} : 0 \leq j \leq 4k - 1\}.$$

Because $\gcd(n, 2k + 1) = 1$, π is a permutation. Similar to the proof of Theorem 4, it is straightforward to show that f is an antipodal labeling, and we shall leave the details to the reader. This completes the proof of Theorem 9. \square

We conjecture that $\text{an}(C_{4k})$ is equal to the upper bound in Theorem 9.

Conjecture 1 For any $k \geq 1$, $\text{an}(C_{4k}) = 2k^2 - 1$.

A case-by-case analysis has confirmed the above conjecture for $k \leq 5$.

References

- [1] T. Calamoneri and R. Petreschi, “ $L(2, 1)$ -labeling of planar graphs,” *ACM* (2001), 28-33.
- [2] G.J. Chang, and C. Lu, “Distance-two labelings of graphs,” *European Journal of Combinatorics*, 24 (2003), 53-58.

- [3] G.J. Chang and D. Kuo, “The $L(2, 1)$ -labeling problem on graphs,” *SIAM J. Discrete Math.*, 9 (1996), 309-316.
- [4] G. Chartrand, D. Erwin, and P. Zhang, “Radio antipodal colorings of cycles,” *Congressus Numerantium*, 144 (2000), 129 – 141.
- [5] G. Chartrand, D. Erwin, and P. Zhang, “Radio antipodal colorings of graphs,” *Math. Bohem.*, 127 (2002), 57 – 69.
- [6] G. Chartrand, D. Erwin, and P. Zhang, “Radio labelings of graphs,” *Bull. Inst. Combin. Appl.*, 33 (2001), 77 – 85.
- [7] G. Chartrand, D. Erwin, and P. Zhang, “Radio k -colorings of paths,” *Discuss Math. Graph Theory*, 24 (2004), 5 – 21.
- [8] G. Chartrand, D. Erwin, and P. Zhang, “A graph labeling problem suggested by FM channel restrictions,” *Bull. Inst. Combin. Appl.*, 43 (2005), 43–57.
- [9] Z. Furedi, J.R. Griggs, and D.J. Kleitman, “Pair labelings with given distance,” *SIAM J. Discrete Math.*, 4 (1989), 491-499.
- [10] J.P. Georges, D.W. Mauro, and M.A. Whittlesey, “Relating path covering to vertex labelings with a condition at distant two,” *Discrete Math.*, 135 (1994), 103-111.
- [11] J.P. Georges, D.W. Mauro, and M.A. Whittlesey, “On the λ -number of Q_n and related graphs,” *SIAM J. Discrete Math.*, 8 (1995), 499-506.
- [12] J.R. Griggs and R. K. Yeh, “Labeling graphs with a condition at distance 2,” *SIAM J. Discrete Math.*, 5 (1992), 586-595.
- [13] R. Khennoufa and O. Tongni, “A note on radio antipodal colourings of paths,” *Math. Bohem.*, 130 (2005), 277 – 282.
- [14] D. D.-F. Liu, and R.K. Yeh, “On distance two labelings of graphs,” *Ars Combin.*, 47 (1997), 13-22.
- [15] D. D.-F. Liu, “Radio number for trees,” manuscript, 2005.
- [16] D. D.-F. Liu and M. Xie, “Radio number for square cycles,” *Cong. Numer.*, 169 (2004), 105 – 125.
- [17] D. D.-F. Liu and M. Xie, “Radio number for square paths,” *Ars Combinatoria*, to appear.
- [18] D. D.-F. Liu and X. Zhu, “Multi-level distance labelings for paths and cycles,” *SIAM Disc. Math.*, 19 (2005), 610 – 621.
- [19] P. Zhang, “Radio Labelings of Cycles,” *Ars. Combin.*, 65 (2002), 21 – 32.