

# K-QUASIDERIVATIONS

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ABSTRACT. A  $K$ -quasiderivation is a map which satisfies both the Product Rule and the Chain Rule. In this paper, we discuss several interesting families of  $K$ -quasiderivations. We first classify all  $K$ -quasiderivations on the ring of polynomials in one variable over an arbitrary commutative ring  $R$  with unity, thereby extending a previous result. In particular, we show that any such  $K$ -quasiderivation must be linear over  $R$ . We then discuss two previously undiscovered collections of (mostly) *nonlinear*  $K$ -quasiderivations on the set of functions defined on some subset of a field. Over the reals, our constructions yield a one-parameter family of  $K$ -quasiderivations which includes the ordinary derivative as a special case.

## 1. INTRODUCTION

In the middle half of the twentieth century—perhaps as a reflection of the mathematical zeitgeist—Lausch, Menger, Müller, Nöbauer and others formulated a general axiomatic framework for the concept of the derivative. Their starting point was (usually) a *composition ring*, by which is meant a commutative ring  $R$  with an additional operation  $\circ$  subject to the restrictions  $(f + g) \circ h = (f \circ h) + (g \circ h)$ ,  $(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$ , and  $(f \circ g) \circ h = f \circ (g \circ h)$  for all  $f, g, h \in R$ . (See [1].) In Müller’s parlance [9], a  *$K$ -derivation* is a map  $D$  from a composition ring to itself such that  $D$  satisfies

$$\text{Additivity: } D(f + g) = D(f) + D(g) \quad (1)$$

$$\text{Product Rule: } D(f \cdot g) = f \cdot D(g) + g \cdot D(f) \quad (2)$$

$$\text{Chain Rule } D(f \circ g) = [(D(f)) \circ g] \cdot D(g) \quad (3)$$

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(The “K” stands for “Kettenregel,” the German word for “Chain Rule.”) It is natural to choose these three as the fundamental properties, since they describe how  $D$  interrelates with the basic operations of addition, multiplication, and composition.

Standard examples of K-derivations on composition rings include the ordinary derivative on the composition ring  $\mathcal{C}^\infty(\mathbb{R})$  of infinitely differentiable functions from the reals to the reals, and formal differentiation on polynomial rings. Other “ $\frac{d}{dx}$ -like” maps that satisfy some or all of (1)–(3), or at least some variants thereof, include partial derivatives, Frechét derivatives, Gâteaux derivatives, quasi-derivatives, Lie derivatives,  $q$ -derivatives,  $H$ -derivatives, Radon-Nikodym derivatives, Pincherle derivatives, Kähler differentials, exterior derivatives, covariant derivatives, and many others. K-derivations have been studied in detail for polynomial rings; rings of rational functions; rings of formal power series with constant term zero; the set of all functions from a ring to itself; and other composition rings. (See, for example, [6], [7], [8], [9], [10], [11], [12], [13].)

One recurring question in these papers is whether it is possible to reduce the number of defining axioms for a K-derivation. For example: Is a  $K$ -quasiderivation (that is, a map which satisfies both the Product Rule and the Chain Rule) necessarily a K-derivation? The answer is: sometimes, but not always. In this paper, we discuss two extreme cases. We first consider polynomial rings. In this setting, the answer is a resounding yes; indeed, we will show that any such K-quasiderivation must equal the ordinary formal derivative times a constant idempotent. We then turn our attention to *partial  $F$ -self-maps*, that is, functions from a subset of a field  $F$  to  $F$ , whereupon we find that, even subject to several additional natural restrictions, K-quasiderivations can be highly nonlinear.

## 2. K-QUASIDERIVATIONS ON POLYNOMIAL RINGS

Recall from page 1 the three properties (1)–(3). In [9], Müller shows that if  $R$  is an integral domain and  $D$  is a map from  $R[x]$  to  $R[x]$  that satisfies (2) and (3), then  $D$  satisfies (1) as well. Using an appropriately modified version of the Chain Rule, Müller obtains the same result for polynomial rings in more than one variable, this time when  $R$  is an arbitrary commutative ring with unity. He attempted, in the one-variable case, to generalize to arbitrary commutative rings with unity but was unable to do so. In this section, we will fill in this last missing piece.

The plan of attack, as in [9], is to classify all maps that satisfy (2) and (3), then note that they all also satisfy (1). Let's first define these maps, and note that they satisfy (1)–(3).

For any  $a \in R$ , define  $d_a : R[x] \rightarrow R[x]$  by  $d_a(f) = a \cdot d(f)$ , where  $d : R[x] \rightarrow R[x]$  is the ordinary formal derivative. Note that for any  $a$ , we have that  $d_a$  satisfies (1) and (2). Moreover, if  $a$  is idempotent, then  $d_a$  satisfies (3) as well.

We can now state the main theorem of this section.

**Theorem 2.1.** *Let  $R$  be a commutative ring with unity. If  $D : R[x] \rightarrow R[x]$  is a  $K$ -quasiderivation, then  $D = d_a$  for some idempotent  $a \in R$ . Hence,  $D$  is linear over  $R$ , and in particular,  $D$  is additive.*

We shall prove this theorem in the subsection below.

Recall that a ring  $R$  is said to be **indecomposable** if it cannot be written as a direct sum of two nonzero rings, and that a ring is indecomposable iff 0 and 1 are the only idempotents. Along the lines of [13], we can therefore use Theorem 2.1 to characterize indecomposable rings, as follows:

**Corollary 2.2.** *Let  $R$  be a commutative ring with unity. Then  $R$  is indecomposable iff the ordinary formal derivative is the only nonzero  $K$ -quasiderivation on  $R[x]$ .*

We remark that neither the Product Rule nor the Chain Rule alone is sufficient to guarantee additivity. The map that sends everything to 1, for example, satisfies the Chain Rule but is nonadditive. The map  $D_3 : \mathbb{Z}_4[x] \rightarrow \mathbb{Z}_4[x]$  defined by

$$D_3(c_0 + \cdots + c_n x^n) = \begin{cases} 0 & \text{if } c_0 \in \{0, 1\} \\ 2 & \text{if } c_0 \in \{2, 3\} \end{cases}$$

satisfies the Product Rule but is nonadditive.

**2.1. Proof of Theorem 2.1.** Let  $R$  be a commutative ring with unity. Let  $R[x]$  be the ring of polynomials with coefficients in  $R$  and indeterminate  $x$ . Let  $D : R[x] \rightarrow R[x]$  be a  $K$ -quasiderivation.

**Lemma 2.3.**  *$D(c) = 0$  for any constant  $c$ .*

*Proof.*  $D(0) = 0$  follows from  $0 \cdot 0 = 0$  and the Product Rule. Then  $D(c) = 0$  follows from  $c \circ 0 = c$  and the Chain Rule.  $\square$

**Lemma 2.4.**  *$D(cf) = cD(f)$  for any constant  $c \in R$  and any polynomial  $f \in R[x]$ .*

*Proof.* Use Lemma 2.3 and the Product Rule.  $\square$

For any  $c \in R$ , let  $\phi_c(x) = x + c \in R[x]$ . Let  $f(x) = D(\phi_0)$ .

**Lemma 2.5.**  $f(x) = a$  for some idempotent  $a \in R$ .

*Proof.* First we show that  $f$  is a constant polynomial. Assume temporarily that it is not. Write  $f(x) = k_0 + k_1x + \cdots + k_nx^n$ .

Apply Chain Rule to  $\phi_0 \circ \phi_0 = \phi_0$  to get:

$$(2.1) \quad f(x) \cdot f(x) = f(x)$$

Comparing the constant terms of both sides of (2.1), we find that  $k_0^2 = k_0$ . Let  $\ell$  be the smallest *positive* integer such that  $k_\ell \neq 0$ . (Since  $f$  is nonconstant, we know that  $\ell$  exists.) Comparing the  $x^\ell$  terms of both sides of (2.1), we find that  $2k_0k_\ell = k_\ell$ . Multiply both sides by  $k_0$  to get that  $2k_0k_\ell = k_0k_\ell$ , which implies that  $k_0k_\ell = 0$ . Hence we have  $0 = 2 \cdot 0 = k_\ell$ , which is a contradiction.

Therefore  $f(x)$  equals a constant  $a \in R$ . Equation (2.1) then shows that  $a$  is idempotent.  $\square$

Let  $a$  be the idempotent in Lemma 2.5.

**Lemma 2.6.** Let  $\psi \in R[x]$ . Then  $D(\psi) = a \cdot D(\psi)$ .

*Proof.* Apply the Chain Rule and Lemma 2.5 to the equation  $\psi \circ \phi_0 = \psi$ .  $\square$

Let  $g(x) = D(\phi_1)$ , and let  $j(x) = D(\phi_{-1})$ .

**Lemma 2.7.**  $j(x) = g(-x)$

*Proof.* Apply the Chain Rule, Lemma 2.4, and Lemma 2.5 to the equation

$$(2.2) \quad \phi_{-1} = -1 \cdot [\phi_1 \circ (-1 \cdot \phi_0)]$$

to get that  $j(x) = a \cdot g(-x)$ . Now apply Lemma 2.6.  $\square$

It may be easier to read equation (2.2) in the form  $x-1 = -((-x)+1)$ , and from now on we will not hesitate to write equations in the latter style, when appropriate.

**Lemma 2.8.**  $g(x) = a$

*Proof.* We first show that  $g$  is a constant polynomial. Temporarily assume otherwise. Write  $g(x) = b_0 + b_1x + \cdots + b_nx^n$ , where  $n > 0$  and  $b_n \neq 0$ .

From the equation  $\phi_{-1} \circ \phi_1 = \phi_0$ , we find that

$$(2.3) \quad g(x) \cdot g(-x-1) = a$$

Comparing the highest order terms of both sides of (2.3), we find that  $b_n^2 = 0$ .

Next apply Lemmas 2.4, 2.5, and 2.7, together with the Product Rule and Chain Rule, to  $a(x^2 - 1) = a(x + 1)(x - 1)$  in order to obtain:

$$2axg(-x^2) = a(x + 1)g(-x) + a(x - 1)g(x)$$

Comparing the highest order terms of both sides, we find that  $2ab_n = 0$ .

Now apply Lemmas 2.4, 2.5, and 2.7, together with Product Rule and Chain Rule, to  $a(x^3 - 1) = a(x - 1) \cdot [x(x + 1) + 1]$  in order to obtain:

$$(2.4) \quad 3ax^2g(-x^3) = a[g(-x) \cdot (x^2 + x + 1) + ax^2g(x^2 + x) - ag(x^2 + x) + x^2g(x^2 + x)g(x) - xg(x^2 + x)g(x)]$$

Comparing the highest order terms of both sides, we find that

$$(-1)^n 3ab_n = ab_n^2.$$

From Lemma 2.6, we have that  $ab_n = b_n$ . Putting the pieces together, we have that

$$b_n = ab_n = 3ab_n - 2ab_n = (-1)^n ab_n^2 - 2ab_n = 0 - 0 = 0,$$

which is a contradiction.

Now that we know  $g$  is a constant, equation (2.4) simplifies to:

$$3agx^2 = a(2g + g^2)x^2 + a(g - g^2)x.$$

From the  $x$  term, we get that  $ag = ag^2$ . Lemma 2.6 and (2.3) then imply that  $g = a$ .  $\square$

**Lemma 2.9.**  $D(\phi_c) = a$  for all  $c \in R$ .

*Proof.* Let  $c \in R$ , and let  $h(x) = D(\phi_c)$ . Applying Lemmas 2.3–2.8 as well as the Product Rule and Chain Rule to the equation  $a(cx + 1)(x^2 + 1) = a(x(x(cx + 1) + c) + 1)$ , we obtain:

$$(2.5) \quad a[3cx^2 + 2x + c] = a[cx^2 + 2cx^2h(cx^2 + x) + x + xh(cx^2 + x) + c]$$

Write  $h(x) = r_0 + r_1x + \cdots + r_nx^n$ . Comparing the  $x$  terms of both sides of (2.5), we find that  $ar_0 = a$ . Comparing the  $x^2$  terms of both sides of (2.5), we find that  $ar_1 = 0$ . The left-hand side of (2.5) has no terms of degree 3 or higher, and so it follows by induction, considering terms of successively higher order, that  $ar_j = 0$  for all  $j \geq 1$ . By Lemma 2.6, we have that  $h(x) = ah(x) = ar_0 + \cdots + ar_nx^n = a$ , as desired.  $\square$

*Proof of Theorem 2.1.* The constant functions and the functions  $\phi_c$  generate  $R[x]$  under multiplication and composition. (For example: In  $\mathbb{Z}[x]$ , we have  $2x^2 + x + 3 = \phi_3 \circ (\phi_0 \cdot (\phi_1 \circ (2 \cdot \phi_0)))$ .) Lemmas 2.3 and 2.9 say that  $D(\psi) = d_a(\psi)$  if  $\psi$  is constant or if  $\psi = \phi_c$  for some  $c \in R$ . Moreover, since  $a$  is idempotent, both  $D$  and  $d_a$  satisfy both the Chain Rule and the Product Rule. So by applying the Chain Rule and Product Rule repeatedly, we find that  $D = d_a$ .  $\square$

### 3. K-QUASIDERIVATIONS AND PARTIAL FIELD SELF-MAPS

In this section, we consider a set which possesses two infinite families of K-quasiderivations which are not K-derivations. Let  $F$  be a field. Nöbauer shows in [13] that the only K-quasiderivation on the composition ring  $F^F$  of all functions from  $F$  to itself is the trivial K-quasiderivation (i.e., the zero map). So, in order to generate interesting families of K-quasiderivations, we relax the restriction that the functions in question must be defined everywhere. This leads us to the concept of a *partial F-self-map*. Let  $\mathcal{M}_F$  be the set of all functions whose domain is some subset of  $F$  and whose codomain is  $F$ . While  $\mathcal{M}_F$  is not a composition ring, it does possess the three operations of addition, multiplication, and composition, and it is moreover a natural space to consider in the context of generalizations of the derivative—after all, the (ordinary) derivative of a function may not be defined everywhere.

The two families we construct in this paper lie at opposite ends of the K-quasiderivation spectrum, in the following sense. We say that a K-quasiderivation  $\mathcal{F}$  is *pointwise at  $x$*  if the value of  $\mathcal{F}f$  at  $x$  is determined by the value of  $f$  at  $x$ . We say that  $\mathcal{F}$  is *pointwise* if  $\mathcal{F}$  is pointwise at all  $x$ , and *nowhere-pointwise* if  $\mathcal{F}$  is pointwise at no  $x$ .

Our first main theorem in this section (Theorem 3.20) classifies all regular pointwise K-quasiderivations. (The regularity condition is imposed to avoid difficulties with domains; it requires that the functions under consideration have inputs and outputs from some multiplicatively closed set.) The classification goes as follows. Let  $\Delta$  be a quasiderivation, i.e. a map which obeys the Product Rule. (The original motivation for this research was an investigation of the quasiderivation  $\Delta_*$ , which is obtained by mapping each prime to 1 and then extending via the Product Rule—see p. 13 for more details.) Define  $\mathcal{F}^\Delta : \mathcal{M}_F \rightarrow \mathcal{M}_F$  by  $(\mathcal{F}^\Delta f)(x) = \frac{\Delta(f(x))}{\Delta(x)}$ . Then every  $\mathcal{F}^\Delta$  is a regular pointwise K-quasiderivation, and every regular pointwise K-quasiderivation equals

$\mathcal{F}^\Delta$  for some  $\Delta$ . To show that this is not vacuous, we provide in §3.3 an example of uncountably many such K-quasiderivations.

Our second main theorem (Theorem 3.15) classifies all rational K-quasiderivations, i.e. K-quasiderivations which eat rational functions. This classification goes as follows. Let  $\mathcal{F}$  be a K-quasiderivation on  $\mathcal{M}_F$ , and let  $a$  be a nonzero element of  $F$ . Define  $\mathcal{F}_{(a)} : \mathcal{M}_F \rightarrow \mathcal{M}_F$  by

$$(\mathcal{F}_{(a)}f)(x) = \begin{cases} a(\mathcal{F}f)(x) & \text{if } x = 0 \text{ and } f(x) \neq 0 \\ \frac{1}{a}(\mathcal{F}f)(x) & \text{if } x \neq 0 \text{ and } f(x) = 0 \\ (\mathcal{F}f)(x) & \text{otherwise.} \end{cases}$$

Then  $\mathcal{F}_{(a)}$  is again a K-quasiderivation. (Proving this requires consideration of sixteen separate cases, and we find it remarkable that they all work out just so. We therefore speculate that there may be a more intrinsic way to view this construction, one which explains why the Product Rule and Chain Rule both hold for  $\mathcal{F}_{(a)}$  without the need for a case-by-case analysis.) Let  $D$  be the ordinary formal derivative on rational functions. Theorem 3.15 states that every  $D_{(a)}$  is a nonzero rational K-quasiderivation, and every nonzero rational K-quasiderivation equals  $D_{(a)}$  for some  $a$ . All of the K-quasiderivations in this family except  $D_{(1)}$  are nonlinear.

One area for future investigation is to attempt to extend the results of this section beyond fields to commutative rings with identity.

**3.1. Basics.** Let  $F$  be a field. In this section, we give some basic definitions and collect some general facts about K-quasiderivations on the set of partial  $F$ -self-maps.

**Definition 3.1.** Let  $\mathcal{M}_F = \{f : S \rightarrow F \mid S \subseteq F\}$ . We call an element of  $\mathcal{M}_F$  a partial  $F$ -self-map.

Note that to define a partial  $F$ -self-map, it is necessary to specify its domain. If  $S \subseteq F$ , then denote by  $f|_S$  the restriction of  $f$  to the intersection of  $S$  with the domain of  $f$ .

We define the basic three operations of addition, multiplication, and composition on  $\mathcal{M}_F$  as follows. Define addition and multiplication pointwise, where the domain of  $f + g$  and  $fg$  is the intersection of the domains of  $f$  and  $g$ . Subtraction and division are defined similarly. Define  $f \circ g$  by  $(f \circ g)(x) = f(g(x))$ , where the domain of  $f \circ g$  is the set of all  $x$  in the domain of  $g$  such that  $g(x)$  is in the domain of  $f$ .

As the elements of  $\mathcal{M}_F$  are functions, we impose one additional restriction on K-quasiderivations. Namely, we insist that they respect restriction to a subset, much like a sheaf map.

**Definition 3.2.** Let  $\mathcal{F} : \mathcal{M}_F \rightarrow \mathcal{M}_F$ . We say that  $\mathcal{F}$  is a K-quasi-derivation on  $\mathcal{M}_F$  if the following axioms hold:

*Chain Rule:* For all  $f, g \in \mathcal{M}_F$ , we have  $\mathcal{F}(f \circ g) = (\mathcal{F}f \circ g) \cdot (\mathcal{F}g)$ .

*Product Rule:* For all  $f, g \in \mathcal{M}_F$ , we have  $\mathcal{F}(f \cdot g) = \mathcal{F}f \cdot g + f \cdot \mathcal{F}g$ .

*Sheaf Map Axiom:* If  $f = g|_S$ , then  $\mathcal{F}f = (\mathcal{F}g)|_S$ .

Note that to be absolutely precise,  $g|_S$  should be read as “ $g$  restricted to the intersection of  $S$  with the domain of  $g$ .”

**Remark.** If a K-quasi-derivation  $\mathcal{F}$  additionally satisfies

Additivity:  $\mathcal{F}(f + g) = \mathcal{F}f + \mathcal{F}g$  for all  $f, g \in \mathcal{M}_F$ ,

then  $\mathcal{F}$  is called a *K-derivation*. The proof of Lemma 3.12 shows that for most K-quasi-derivations, additivity is equivalent to linearity over  $F$ .

For all  $c \in F$ , let  $\bar{c}$  denote the constant function  $c$  with domain  $F$ . Throughout the rest of this section, let  $\mathcal{F}$  be a K-quasi-derivation on  $\mathcal{M}_F$ .

**Lemma 3.3.** Let  $\lambda \in \{-1, 0, 1\}$ , and let  $S \subseteq F$ . Then  $(\mathcal{F}\bar{\lambda}|_S)(x) = 0$  for all  $x$  in the intersection of  $S$  with the domain of  $\mathcal{F}\bar{\lambda}$ .

*Proof.* For any  $\lambda \in F$ , the Product Rule and the Sheaf Map Axiom imply that  $\mathcal{F}\bar{\lambda}^2|_S = 2\bar{\lambda}\mathcal{F}\bar{\lambda}|_S$ . The lemma follows immediately upon considering this equation with  $\lambda = 1, -1$ , and  $0$  respectively.  $\square$

One must be careful. Although K-quasi-derivations are generalizations of derivatives,  $\mathcal{F}\bar{\lambda}$  may very well be nonzero if  $\lambda \notin \{-1, 0, 1\}$  and  $\mathcal{F}$  is nonlinear! The regular pointwise K-quasi-derivations of §3.3 provide many examples of this phenomenon.

**Lemma 3.4.** Let  $i_S(x) = x$  be the identity function with domain  $S$ . Then for all  $x_0$  in the domain of  $\mathcal{F}i_S$ , we have  $(\mathcal{F}i_S)(x_0) \in \{0, 1\}$ . Moreover, if  $x_1 \in F$  and  $g(x_1)$  is in the domain of  $\mathcal{F}i_F$  and  $(\mathcal{F}g)(x_1) \neq 0$ , then  $(\mathcal{F}i_F)(g(x_1)) = 1$ .

*Proof.* Apply the Chain Rule to the equation  $i_S \circ i_S = i_S$  to conclude that  $[(\mathcal{F}i_S)(x_0)]^2 = (\mathcal{F}i_S)(x_0)$ , hence  $(\mathcal{F}i_S)(x_0) = 0$  or  $(\mathcal{F}i_S)(x_0) = 1$ . For the second part, taking the derivative of  $i_F \circ g = g$  yields

$$(\mathcal{F}i_F)(g(x_1)) \cdot (\mathcal{F}g)(x_1) = (\mathcal{F}g)(x_1),$$

and one may divide by  $(\mathcal{F}g)(x_1)$ .  $\square$

Following the example from his or her calculus class, the reader may see immediately how to prove the following three lemmas.

**Lemma 3.5** (The Power Rule). *Let  $f \in \mathcal{M}_F$ , and let  $\nu \in \mathbb{Z}$  such that  $f^\nu \in \mathcal{M}_F$ . Then*

$$\mathcal{F}(f^\nu) = \nu f^{\nu-1} \mathcal{F}f.$$

**Lemma 3.6** (The Quotient Rule). *Let  $f, g \in \mathcal{M}_F$ . Then*

$$\mathcal{F}\left(\frac{f}{g}\right) = \frac{g \cdot \mathcal{F}f - f \cdot \mathcal{F}g}{g^2}.$$

**Lemma 3.7** (The Inverse Function Rule). *If  $f$  and  $g$  are inverse functions, then for  $x_0$  in the domain of  $\mathcal{F}g$  either  $\mathcal{F}g(x_0) = 0$  or  $\mathcal{F}g(x_0) = [\mathcal{F}f \circ g(x_0)]^{-1}$ .*

**Lemma 3.8.** *If  $F$  is a finite field and  $f \in \mathcal{M}_F$ , then  $(\mathcal{F}f)(x) = 0$  for all  $x$  in the domain of  $\mathcal{F}f$ .*

*Proof.* Let  $q = |F|$  and apply the Power Rule to the equation  $f^q = f$ .  $\square$

Let  $F^\times$  be the set of nonzero elements of  $F$ .

We have a general method for constructing new K-quasiderivations from old ones, namely the following.

**Definition 3.9.** *Let  $a \in F^\times$ . Define  $\mathcal{F}_{(a)}$  by:*

$$(\mathcal{F}_{(a)}f)(x) = \begin{cases} a(\mathcal{F}f)(x) & \text{if } x = 0 \text{ and } f(x) \neq 0 \\ \frac{1}{a}(\mathcal{F}f)(x) & \text{if } x \neq 0 \text{ and } f(x) = 0 \\ (\mathcal{F}f)(x) & \text{otherwise.} \end{cases}$$

Note that it is implicit in Definition 3.9 that the domain of  $\mathcal{F}_{(a)}f$  equals the domain of  $\mathcal{F}f$ .

**Lemma 3.10.** *If  $\mathcal{F}$  is a K-quasiderivation on  $\mathcal{M}_F$  and  $a \in F^\times$ , then  $\mathcal{F}_{(a)}$  is also a K-quasiderivation on  $\mathcal{M}_F$ .*

*Proof.* The proof is straightforward but consists of checking sixteen separate cases and so is omitted here. (There are eight cases for the Product Rule and eight for the Chain Rule, according to whether  $x$ ,  $f(x)$ , and  $g(x)$  are 0. Some of these cases are vacuous.)  $\square$

**3.2. Rational K-quasiderivations.** In this section, we define the notion of a ‘‘rational’’ K-quasiderivation and classify all rational K-quasiderivations. These K-quasiderivations are in a sense at the opposite end of the K-quasiderivation spectrum from the regular pointwise K-quasiderivations, insofar as every nonzero rational K-quasiderivation is nowhere pointwise. We find that for an infinite field  $F$ , there is a

1-1 correspondence between elements of  $F$  and rational  $K$ -quasiderivations on  $F$ . As all but two of these  $K$ -quasiderivations are nonadditive, we therefore once again have an infinite family of  $K$ -quasiderivations which are not  $K$ -derivations.

Because of Lemma 3.8, we will assume through this section that  $F$  is infinite. Since  $F$  is an infinite field, the composition field  $F(x)$  of rational functions with coefficients in  $F$  embeds in  $\mathcal{M}_F$ . This allows us to make the following definition.

**Definition 3.11.** *Let  $\mathcal{F}$  be a  $K$ -quasiderivation on  $F$ . We say that  $\mathcal{F}$  is rational if the domain of  $\mathcal{F}f$  equals the domain of  $f$  whenever  $f = g|_S$  for some rational function  $g$  and cofinite set  $S$ , and  $\mathcal{F}f$  has empty domain otherwise.*

Throughout the rest of this section, let  $\mathcal{F}$  denote a rational  $K$ -quasiderivation on  $\mathcal{M}_F$ .

**Lemma 3.12.** *Let  $c \in F$ , and let  $f \in F(x)$ . Then:*

- a.  $\mathcal{F}\bar{c} = \bar{0}$ .
- b.  $\mathcal{F}(\bar{c} \cdot f) = \bar{c} \cdot (\mathcal{F}f)$

*Proof.* a. Apply the Chain Rule and Lemma 3.3 to the equation  $\bar{c} \circ \bar{0} = \bar{c}$ .

b. Use the Product Rule and (a). □

For any  $c \in F$ , define the function  $\phi_c$  by  $\phi_c(x) = x + c$ . Let  $j = \mathcal{F}\phi_0$ , and let  $k = \mathcal{F}\phi_1$ .

**Lemma 3.13.** *Let  $c \in F^\times$ . Then  $(\mathcal{F}\phi_c)(x) = k(\frac{x}{c})j(x)$ .*

*Proof.* Apply the Chain Rule and Lemma 3.12 to the equation

$$(3.1) \quad \phi_c = \bar{c} \cdot (\phi_1 \circ (\bar{c}^{-1} \cdot \phi_0)). \quad \square$$

It may be easier to read equation (3.1) in the somewhat less precise form  $x + c = c(\frac{x}{c} + 1)$ , and from now on we will not hesitate to write equations in the latter style, when appropriate.

**Lemma 3.14.**  *$j = \bar{0}$  or  $j = \bar{1}$ .*

*Proof.* Suppose  $j \neq \bar{0}$ . Let  $x_0 \in F$  such that  $j(x_0) \neq 0$ . Applying the Chain Rule and Lemma 3.13 to the equation  $\phi_{-c} \circ \phi_c = \phi_0$  for any nonzero  $c$  and then evaluating at  $x_0$ , we find that  $j(x) \neq 0$  for all  $x \in F$ . The result then follows from Lemma 3.4. □

Let  $D$  be the ordinary formal derivative on  $F(x)$ . Then  $D$  defines a rational  $K$ -quasiderivation on  $\mathcal{M}_F$ . Our next theorem states that every nonzero rational  $K$ -quasiderivation on  $\mathcal{M}_F$  is induced from  $D$  via Definition 3.9.

**Theorem 3.15.** *If  $a \in F^\times$ , then  $D_{(a)}$  is a rational K-quasiderivation on  $\mathcal{M}_F$ . Conversely, if  $\mathcal{F}$  is a rational K-quasiderivation on  $\mathcal{M}_F$ , then either  $\mathcal{F}$  is the zero map or else  $\mathcal{F} = D_{(a)}$  for some  $a \in F^\times$ .*

*Proof.* It is straightforward to verify that  $D_{(a)}$  is a rational K-quasiderivation on  $\mathcal{M}_F$ . Let  $\mathcal{F}$  be a rational K-quasiderivation on  $\mathcal{M}_F$ . If  $\mathcal{F}\phi_0 = \bar{0}$ , then  $\mathcal{F}$  is the zero map, which we can see by applying the Chain Rule to the equation  $f \circ \phi_0 = f$  for all  $f$ .

So by Lemma 3.14, we have that  $j = \mathcal{F}\phi_0 = \bar{1}$ . Let  $\psi(x) = x^{-1}$ . It follows from the Product Rule, the equation  $\psi \cdot \phi_0 = \bar{1}|_{F^\times}$ , and Lemma 3.3 that  $(\mathcal{F}\psi)(x) = -x^{-2}$ .

Let  $k = \mathcal{F}\phi_1$ . Applying Lemma 3.13 and the Chain Rule to the equation  $\phi_{-1} \circ \phi_1 = \phi_0$  and then evaluating at 0, we find that

$$(3.2) \quad k(-1)k(0) = 1.$$

Next, for any  $c \notin \{0, -1\}$ , let  $b = -\frac{c}{c+1}$ , apply Lemma 3.13 and the Chain Rule to the equation  $\phi_1 \circ \phi_b = \phi_{1+b}$ , and evaluate at  $-b$  to find that:

$$(3.3) \quad \text{If } c \neq 0 \text{ and } c \neq -1, \text{ then } k(c) = 1.$$

Let  $a = k(0)$ . Lemma 3.12 shows that  $D_{(a)}(\bar{c}) = \mathcal{F}(\bar{c})$  for all constants  $c$ . Moreover, we have that  $D_{(a)}x = \bar{1} = \mathcal{F}x$  and  $D_{(a)}(x^{-1}) = -x^{-2} = (\mathcal{F}\psi)(x^{-1})$ . Equations (3.2) and (3.3) show that  $D_{(a)}(x+1) = \mathcal{F}(x+1)$ . But  $F(x)$  is generated, under multiplication and composition, by the constant functions and the functions  $x$ ,  $x+1$ , and  $x^{-1}$ . Since  $D_{(a)}$  coincides with  $\mathcal{F}$  on all of these functions, and since both  $D_{(a)}$  and  $\mathcal{F}$  obey both the Product Rule and the Chain Rule, we find that  $D_{(a)} = \mathcal{F}$ .  $\square$

**Remark.**  $D_{(a)}$  is “nowhere-pointwise,” in the sense that it is not pointwise at  $x$  for any  $x \in F$ .

**Remark.**  $D_{(a)}$  is additive iff  $a = 1$ . So for any infinite field  $F$ , we have an infinite family of nonlinear K-quasiderivations. While  $D_{(a)}$  is in general nonadditive, for any rational function  $f$  we do have that  $D_{(a)}f$  coincides with  $Df$  outside the zero set of  $f$ . So, we might say that a rational K-quasiderivation is “cofinitely linear.”

**Remark.** Define a partial ordering on the set of all K-quasiderivations on  $\mathcal{M}_F$  as follows. We say that  $\mathcal{F} < \mathcal{G}$  if for all  $f \in \mathcal{M}_F$  we have that  $\mathcal{F}f$  is a restriction of  $\mathcal{G}f$ . In the case where  $F$  is the field  $\mathbb{R}$  of real numbers,  $D$  can of course be extended to include not only rational functions but all differentiable functions. More precisely, for any  $f \in \mathcal{M}_{\mathbb{R}}$ , define  $Df$  by  $(Df)(x) = f'(x)$ , where the domain of  $Df$

is the set of all  $x \in \mathbb{R}$  such that  $x$  is a limit point of the domain of  $f$  and  $f$  can be extended so as to be differentiable at  $x$ . We may then ask: Are the  $K$ -quasiderivations  $D_{(a)}$  maximal with respect to this partial ordering?

One can play a similar game and ask a similar question about  $\ell$ -adic completions of  $\mathbb{Q}$ .

**3.3. Pointwise  $K$ -quasiderivations.** In this section, we define the notions of “pointwise” and “regular pointwise”  $K$ -quasiderivations, classify all regular pointwise  $K$ -quasiderivations, and give an example of an infinite family of regular pointwise  $K$ -quasiderivations, none of which are  $K$ -derivations.

**Definition 3.16.** *A  $K$ -quasiderivation  $\mathcal{F}$  is called pointwise at  $x_0$  if  $(\mathcal{F}f)(x_0) = (\mathcal{F}g)(x_0)$  whenever  $f(x_0) = g(x_0)$ . We say  $\mathcal{F}$  is pointwise if it is pointwise at every point  $x_0 \in F$ .*

**Definition 3.17.** *A  $K$ -quasiderivation  $\mathcal{F}$  is called regular pointwise if  $\mathcal{F}$  is pointwise and there exists  $M \subseteq F^\times$  such that  $M$  is nonempty and closed under multiplication and for all  $x \in F$  and  $f \in \mathcal{M}_F$ , we have that  $(\mathcal{F}f)(x)$  is defined iff  $x, f(x) \in M$ . Slightly abusing the standard terminology, we say that  $M$  is the domain of  $\mathcal{F}$ .*

Identifying  $\mathcal{F}$  with the map  $\mathcal{F}(x, y) = (\mathcal{F}\bar{y})(x)$ , we see that a regular pointwise  $K$ -quasiderivation with domain  $M$  is equivalent to a map  $\mathcal{F} : M \times M \rightarrow F^\times$  subject to the requirement that

$$(3.4) \quad \mathcal{F}(x, yz) = y \cdot \mathcal{F}(x, z) + z \cdot \mathcal{F}(x, y)$$

and

$$(3.5) \quad \mathcal{F}(x, z) = \mathcal{F}(y, z) \cdot \mathcal{F}(x, y)$$

for all  $x, y, z \in M$ .

Henceforth we shall write regular pointwise  $K$ -quasiderivations in this form.

**Definition 3.18.** *A quasiderivation from  $M$  to  $F^\times$  is a map  $\Delta : M \rightarrow F^\times$  such that  $\Delta(ab) = a \cdot \Delta(b) + b \cdot \Delta(a)$  for all  $a, b \in M$ .*

**Definition 3.19.** *Let  $\Delta$  be a quasiderivation from  $M$  to  $F^\times$ . Define  $\mathcal{F}^\Delta : M \times M \rightarrow F^\times$  by  $\mathcal{F}^\Delta(x, y) = \frac{\Delta(y)}{\Delta(x)}$ .*

**Theorem 3.20.** *Let  $\Delta$  be a quasiderivation from  $M$  to  $F^\times$ . Then  $\mathcal{F}^\Delta$  defines a regular pointwise  $K$ -quasiderivation. Conversely, if  $\mathcal{F}$  is a regular pointwise  $K$ -quasiderivation with domain  $M$ , then  $\mathcal{F} = \mathcal{F}^\Delta$  for some quasiderivation  $\Delta$  from  $M$  to  $F^\times$ .*

*Proof.* It is straightforward to verify that if  $\Delta$  is a quasiderivation from  $M$  to  $F^\times$ , then  $\mathcal{F}^\Delta$  satisfies (3.4) and (3.5). For the converse, let  $x_0 \in M$  and let  $\Delta(y) = \mathcal{F}(x_0, y)$ . Then (3.4) implies that  $\Delta$  is a quasiderivation from  $M$  to  $F^\times$ , and (3.5) implies that  $\mathcal{F} = \mathcal{F}^\Delta$ .  $\square$

**Remark.** We find it intriguing that the expression  $\frac{\Delta(y)}{\Delta(x)}$  bears a formal resemblance to the difference quotient.

Theorem 3.20 reduces the problem of finding regular pointwise K-quasiderivations on  $\mathcal{M}_F$  to that of finding quasiderivations on  $F$ , which in turn reduces to a question about generators and relations in  $F^\times$ . The latter question is particularly tractable when  $F$  is the field of rational numbers, for the Fundamental Theorem of Arithmetic asserts that the generators are the primes, and that there are no relations between them.

**Remark.** Let  $\mathbb{Q}^+$  denote the set of positive rational numbers, let  $\mathcal{P} = \{2, 3, 5, \dots\}$  denote the set of positive prime numbers, and let  $\Delta : \mathcal{P} \rightarrow \mathbb{Q}^+$ . Extend  $\Delta$  to a function from  $\mathbb{Q}^+$  to  $\mathbb{Q}^+$  by:

$$\Delta \left( \prod_{p \in \mathcal{P}} p^{v_p} \right) = \left[ \sum_{p \in \mathcal{P}} \frac{\Delta(p)}{p} v_p \right] \cdot \left( \prod_{p \in \mathcal{P}} p^{v_p} \right),$$

where each  $v_p$  is an integer.

Let  $M = \{x \in \mathbb{Q}^+ \mid \Delta(x) > 0\}$ . Note that  $M$  contains all integers greater than 1 and hence is nonempty. It is straightforward to verify that  $\Delta$  defines a quasiderivation from  $M$  to  $\mathbb{Q}^\times$ . (The fact that  $\Delta$  satisfies the product rule implies that  $M$  is closed under multiplication.) By Theorem 3.20, we therefore obtain in this manner an infinite (in fact, uncountable) family of K-quasiderivations. We remark that  $\mathcal{F}^\Delta$  is non-additive and is therefore a K-quasiderivation but not a K-derivation.

**Remark.** Let  $\Delta_\star$  be the quasiderivation with  $\Delta_\star(p) = 1$  for all primes  $p$ . On the first few natural numbers, the values of  $\Delta_\star$  are given in the following table.

$n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\Delta_\star(n)$	0	1	1	4	1	5	1	12	6	7	1	16	1	9	8	32

See the On-Line Encyclopedia of Integer Sequences [A003415]. The function  $\Delta_\star$  also appeared in the Putnam Competition [5, page 295].

It is an interesting and possibly quite difficult question as to whether  $\Delta_\star$  is surjective.

**Example.**  $\Delta_\star$  gives an example of a K-quasiderivation where constants do not map to zero.

**Remark.** Quasiderivations, a.k.a. number derivatives, have been studied in [2], [4], [14], [15], and [16], with particular attention paid

to the function  $\Delta_*$ . The treatment in [15] is a particularly good introduction and shows the connections between number derivatives and such famous open problems as the Goldbach and Twin Primes conjectures.

**Remark.** Quasiderivations on the ring  $\mathbb{Z}_n$  of integers modulo  $n$  are classified in [3].

**Remark.** It would be interesting to classify all quasiderivations on an algebraic number field  $K$ . While this is trivial if  $K$  is the field of fractions of a UFD, it may be less so in general.

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