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On the Convergence of Blind Channel Equalization

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Abstract: *Baud-rate blind equalization algorithms may converge to undesirable stable equilibria due to different reasons. One is the use of FIR filter as an equalizer. It is proved in this paper that this kind of local minima exist for all blind equalization algorithms. The local minima generated by this mechanism are thus called* unavoidable local minima. *The other one is due to the cost function adopted by the blind algorithm itself, which has local minima even implemented with double infinite length equalizers. This type of local minima are called* inherent local minima. *It is also shown that the Godard algorithms [10] and standard cumulant algorithms [6] have no inherent local minimum. However, other algorithms, such as the decision-directed equalizer and the Stop-and-Go algorithm [17], have inherent local minima. This paper also studies the convergence of the Godard algorithms [10] and standard cumulant algorithms [6] under Gaussian noise, and derives the mean square error of the equalizer at the global minimum point. The analysis results are confirmed by computer simulations.*

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1 Introduction

Blind channel equalization is an effective tool in digital communication systems to remove the inter-symbol interference (ISI), especially in the situation where the training sequence is costly or impractical. Blind channel equalization algorithms are usually designed to minimize cost functions formed from statistics of channel output signals. Therefore, if, under some circumstances, the cost function of an algorithm has local minima in addition to the global ones, local convergence may occur.

Local convergence of blind equalization algorithms may be a result of two different causes. One is due to the finite length of the equalizer filter. As will be proved in this paper, this kind of local minima exist for every blind equalization algorithm operating on the baud-rate sampled channel output. They are thus called *unavoidable local minima.* Another kind of local minima are due to the poor selection of cost functions, which can have local minima even with double infinite length equalizers. Local minima generated by this mechanism are called *inherent local minima.* This kind of local minima can be prevented by choosing well-designed cost functions.

The Sato algorithm (SA) [18] is the first blind equalization algorithm, which was generalized into a set of BGR algorithms (BGRA) [1]. It has been proved in [5] that both unavoidable and inherent local minima exist for SA and the BGRA [5]. The Godard algorithm (GA) [10] and Shalvi-Weinstein algorithm (SWA) [24, 25] are two of well-known and effective algorithms. It is shown in [3, 4, 7] that Godard cost function also has unavoidable local minima, but it has no inherent local minima. The result in [12] has proved that there is one-to-one correspondence between the minima of the GA and SWA. Therefore, SWA has the same convergence performance as GA does. The unavoidable local convergence of the decision directed equalizer (DDE) and the Stop-and-Go algorithm (SGA) [17], has been analyzed in [11, 14, 15]. For channels with binary inputs, the DDE and SGA are identical to the SA, whose inherent convergence is investigated in [5].

From convergence analyses of [3, 4, 5, 12, 14, 15], unavoidable local minima exist for BGRA, SA, GA, SWA, and DDE algorithms. Therefore, an interesting question is whether unavoidable local minima exist for every blind equalization algorithm. The convergence analyses of blind algorithms mentioned above are based on the assumption that there is no channel noise. But, the channel noise always exists in physical communication systems even though it may be very small in most cases. In fact, there have been some expectations that channel noises may help equalizer parameters to escape some shallow local minima. Therefore, the convergence behavior of blind equalization algorithms under the noise effect is also an important problem. This paper will address these two problems.

The main part of this paper is organized as following. Section 2 briefly introduces the blind equalization in PAM digital communication systems. In Section 3, we will prove that the unavoidable minima exist for every blind equalization algorithm. Then, in Section 4, we will show that standard cumulant algorithm (SCA) has no inherent local minima. We also study the global convergence behavior of GA and SCA under white Gaussian noise, and derive the mean square error (MSE) of the system output upon converging to the global minima. In Section 5, we will demonstrate that inherent local minima exist for DDE and SGA when the channel inputs are 4-level PAM signals. Our analysis results are confirmed by computer simulations in Section 5.

2 Blind Equalization

A. Equalization in digital communication systems

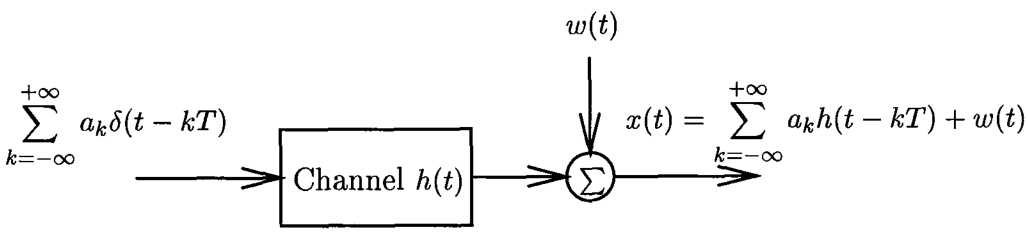

Figure 1: Baseband representation of PAM communication system

Without loss of generality, we consider a baseband representation of the pulse-amplitude-modulation (PAM) communication system as shown in Figure 1. A sequence of independent, identically distributed (i.i.d.) digital signal $\{a_n \in \mathcal{R}\}$ is sent by the transmitter at the symbol rate of $1/T$ through a channel exhibiting linear distortion. The resulting output signal $x(t)$ can be expressed as

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k h(t - kT) + w(t) \tag{2.1}$$

where $w(t)$ is white Gaussian channel noise and $h(t)$ is the impulse response of the linear time-invariant (LTI) channel. If the channel output is sampled at the baud rate $1/T$, a stationary sequence is obtained, which can be expressed as

$$x_n \triangleq x(nT) = \sum_{k=-\infty}^{+\infty} a_{n-k} h_k + w_n, \tag{2.2}$$

in which we have used definitions

$$h_n \triangleq h(nT) \qquad \text{and} \qquad w_n \triangleq w(nT). \tag{2.3}$$

We will assume that the equivalent discrete channel

$$H(\omega) \triangleq \sum_n h_n e^{-jn\omega}$$

is bounded-input-bounded-output (BIBO) stable, which implies that

$$\sum_n |h_n| < \infty. \tag{2.4}$$

A linear channel equalizer

$$\Theta(\omega) \triangleq \sum_n \theta_n e^{-jn\omega}$$

is applied to the channel output $\{x_n\}$ in order to eliminate the ISI. For the equalizer to be BIBO, it must satisfy the condition

$$\sum_n |\theta_n| < \infty. \tag{2.5}$$

The equalizer parameters $\{\theta_n\}$ are subject to adaptation via some algorithm to be determined. If we let

$$S(\omega) \triangleq H(\omega)\Theta(\omega) = \sum_n s_n e^{-jn\omega}, \tag{2.6}$$

the equalizer output as shown in Figure 2 can also be written as

$$\begin{aligned} y_n &= \sum_{k=-\infty}^{\infty} \theta_k x_{n-k} \\ &= \sum_{k=-\infty}^{\infty} s_k a_{n-k}, \end{aligned} \tag{2.7}$$

where

$$s_n \triangleq \sum_k h_k \theta_{n-k}. \tag{2.8}$$

From (2.4) and (2.5), it is obvious that $S(\omega)$ is also BIBO since

$$\sum_n |s_n| < \infty. \tag{2.9}$$

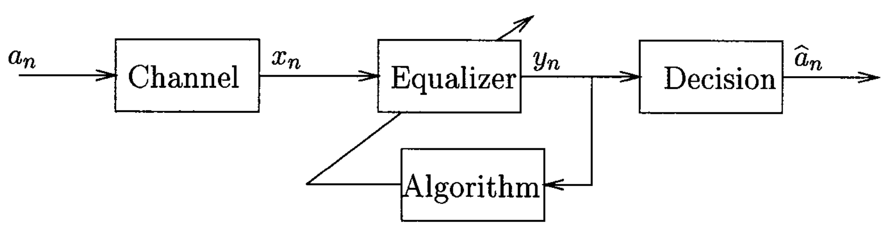

Figure 2: Diagram of typical channel equalization system.

In blind equalization, the original sequence is unknown to the receiver except for its probabilistic or statistical properties over the known alphabet $\mathcal{A} \subset \mathcal{R}$ ($\mathcal{R}$ denotes the set of real number here). Usually, this signal constellation $\mathcal{A}$ is symmetric. Should this be the case, the statistics of the i.i.d. input data reflect the same symmetry. Thus, the recoverable data from blind equalization will be similarly subjected to a sign ambiguity and the best possible result is

$$H(\omega)\Theta(\omega) = \pm e^{-jn_d\omega}, \tag{2.10}$$

for some integer n_d.

B. Blind equalization algorithms

A blind equalization algorithm is usually devised by minimizing a cost function consisting of the statistics of the output of the equalizer y_n.Hence, the cost function is a function of $\{\cdots,\ s_{-1},\ s_0,\ s_1,\cdots\}$ or $\cdots,\ \theta_{-1},\ \theta_0,\ \theta_1,\cdots$, and its global minimum points are

$$\{s_n\} = \pm\{\delta[n - n_d]\} \quad \text{for all} \ \ n_d = 0, \pm 1, \pm 2, \cdots \tag{2.11}$$

to attain perfect equalization (2.10). The Godard algorithm (GA) [10], also known as the constant modulus algorithm (CMA), is one of the widely used algorithms. The algorithm adjusts the equalizer parameter $\{\theta_n\}$ by minimizing the cost function

$$f_{GA}(\theta) \triangleq \frac{1}{4}E\{(|y_n|^2 - \gamma)^2\}, \quad \gamma = \frac{E\{|a_n|^4\}}{E\{|a_n|^2\}}. \tag{2.12}$$

Let $C^p_{y_n}$ be the p-th order cumulant of y_n defined as

$$C^p_y \triangleq (-\jmath\frac{d}{dt})^p \log E\{e^{\jmath t y}\}\ |_{t=0}\ . \tag{2.13}$$

The standard cumulant algorithms (SCA) [6] are defined by minimizing the cost function

$$f_{SCA}(\theta) \triangleq -|C^{2m}_{y_n}| \tag{2.14}$$

for some integer $m > 1$ subject to $|C^2_{y_n}| = |C^2_{a_n}|$. As explained in [25], the Shalvi-Weinstein algorithm (SWA) [24] is a special case ($m = 2$) of the SCA.

If stochastic gradient descent method is used to minimize the cost function, an on-line adaptive equalization algorithm is obtained to adjust the parameters of the equalizer via

$$\widehat{\theta}^{(n+1)}_k = \widehat{\theta}^{(n)}_k - \mu\psi(y_n)x_{n-k}, \tag{2.15}$$

where μ is a small step size. Note that the function $\psi(.)$ relates to the cost function through

$$f(\theta) = E\{\Psi(y_n)\}, \qquad \Psi(y_n) = \int_0^{y_n} \psi(x)dx, \tag{2.16}$$

and $\psi(.)$ is sometimes refered to as *error function.*

Some blind equalization algorithms are directly defined by (2.15). Let

$$\psi(y_n) = w(y_n)(y_n - \widehat{y}_n), \tag{2.17}$$

If $w(y_n) = 1$ for all y_n in (2.17), then (2.15) is the decision-directed equalizer (DDE) [15]. If $w(y_n)$ is defined as the following

$$w(y_n) = \begin{cases} 1 & \text{if } \operatorname{sign}(y_n - \widehat{y}_n) = \operatorname{sign}(y_n - \beta\operatorname{sign}(y_n)), \\ 0 & \text{otherwise} \end{cases} \quad \text{for some } \beta > 0 \tag{2.18}$$

the algorithm defined by (2.15), (2.17) and (2.18) is the Stop-and-Go algorithm (SGA) [17]. Besides the DDE and the SGA, the BGR algorithms (BGRA) [1] and the Sato algorithm (SA) [18] can also be directly defined by (2.15).

3 Unavoidable Local Minima of the Blind Equalization Algorithms

If an FIR filter is used as the equalizer, as is often the case in communication systems, the cost function may have undesirable minima (or local convergence). Undesirable local convergence behavior has been shown for the Godard algorithm (CMA case) [3, 4], the BGR algorithm[5], and the SW algorithm[12]. We will prove in this section that all blind equalization algorithms have local minima if they are implemented with FIR filters.

We first introduce several useful notations and definitions. Let $\ell^1(\mathcal{R})$ denotes the set of all real sequences $\mathbf{u} = \{\cdots,\ u_{-1},\ u_0,\ u_1, \cdots\}$ with finite ℓ^1-norm, i.e.,

$$\| \mathbf{u} \| = \sum_{n=-\infty}^{\infty} |u_n| < \infty. \tag{3.1}$$

From (2.4), (2.5), and (2.9), it is obvious that

$$\mathbf{h} \triangleq (\cdots, h_{-1}, h_0, h_1, \cdots) \in \ell^1(\mathcal{R}), \tag{3.2}$$

$$\theta \triangleq (\cdots, \theta_{-1}, \theta_0, \theta_1, \cdots) \in \ell^1(\mathcal{R}), \tag{3.3}$$

and

$$\mathbf{s} \triangleq (\cdots, s_{-1}, s_0, s_1, \cdots) \in \ell^1(\mathcal{R}). \tag{3.4}$$

The cost function can be written as $f(\mathbf{s})$ or $g(\theta)$, whcih is a functional on $\ell^1(\mathcal{R})$.

The *Super-ball* $\Phi(\mathbf{o},\ \rho)$ with center $\mathbf{o}$ and radius ρ is defined as

$$\Phi(\mathbf{o},\ \rho) = \{\mathbf{s} \in \ell^1(\mathcal{R}) : \|\mathbf{s} - \mathbf{o}\| \leq \rho\}. \tag{3.5}$$

The *Unique global minimum cones* are defined as

$$\begin{aligned} S_n^+ &\triangleq \{\mathbf{s} \in \ell^1(\mathcal{R}) : s_n > 0 \text{ and } s_n > |s_k| \text{ for all } k \neq n\} \\ S_n^- &\triangleq \{\mathbf{s} : -\mathbf{s} \in S_n^+\} \end{aligned} \tag{3.6}$$

Their boundaries are respectively given by

$$\begin{aligned} B_n^+ &\triangleq \{\mathbf{s} \in \ell^1(\mathcal{R}) : s_n > 0 \text{ and } s_n \geq |s_k| \text{ and the equality holds for some } k \neq n\} \\ B_n^- &\triangleq \{\mathbf{s} : -\mathbf{s} \in B_n^+\} \end{aligned} \tag{3.7}$$

In S_n^+ (or S_n^-) the cost function $f(\mathbf{s})$ has only one global minimum point

$$\mathbf{e}_n^+ \triangleq (\cdots,\ 0,\ \underbrace{1}_{\text{(n-th)}},\ 0,\cdots), \tag{3.8}$$

(or $\mathbf{e}_n^- = -\mathbf{e}_n^+$).

With the above definitions, we can demonstrate the local convergence of blind equalization algorithms. We will assume that the cost function of the discussed blind equalization algorithms satisfy the following two conditions:

C1: $\mathbf{e}_n^+$ and $\mathbf{e}_n^-$ for $n = \cdots,\ -1,\ 0,\ 1,\cdots$ are the only global minimum points of the cost function $f(\mathbf{s})$, and

C2: the cost function $f(\mathbf{s})$ is continuous on $\ell^1(\mathcal{R})$.

Denote

$$f_{mb} \triangleq \min_{\mathbf{s}\in B_n^+} f(\mathbf{s}). \tag{3.9}$$

According to condition C1,

$$f_{mb} > f(\mathbf{e}_n^+). \tag{3.10}$$

Therefore, from Condition C2

$$O_n^+ \triangleq \{\mathbf{s} \in S_n^+ : f(\mathbf{s}) < f_{mb}\} \text{ for } n = \cdots,\ -1,\ 0,\ 1,\cdots \tag{3.11}$$

is an open set in $\ell^1(\mathcal{R})$ containing $\mathbf{e}_n^+$ since $f(\mathbf{s})$ is continuous. Hence, there exists a $\rho > 0$ such that

$$\Phi(\mathbf{e}_n^+,\ \rho) \in O_n^+. \tag{3.12}$$

It is obvious that

$$\max_{\mathbf{s}\in\Phi(\mathbf{e}_n^+,\ \rho)} f(\mathbf{s}) < f_{mb}. \tag{3.13}$$

A finite-length equalizer filter

$$\sum_{n=N_1}^{N_2} \theta_n z^{-n}$$

with coefficients θ_n is normally used as an equalizer in practical communication systems, which means only these $N_1 + N_2 + 1$ coefficients can be adjusted and the rest of them are fixed to be zero. Hence, not all $\mathbf{s} \in \ell^1(\mathcal{R})$ can be attained by the equalizer and the *Attainable set* A is

$$A \triangleq \{\mathbf{s} \in \ell^1(\mathcal{R}) : s_n = \sum_{k=N_1}^{N_2} \theta_k h_{n-k}, \theta_k \in \mathcal{R}\}. \tag{3.14}$$

By restricting the algorithm minimization of of $f(\mathbf{s})$ on A, local minimum points can be generated. The proof is given here.

Let the channel impulse response of a PAM communication system be

$$h_n = \alpha^n u[n], \tag{3.15}$$

where $u[n]$ is the unit step function and $\alpha = \frac{\rho}{1+\rho}$. It is a first-order auto-regressive system, hence, a first-order moving-average system can perfectly equalize the channel. If an N-tap filter with coefficients θ_0, $\theta_1, \cdots$, θ_{N-1} is used as equalizer, the attainable set can be expressed as

$$A = \{\mathbf{s} : s_n = \sum_{k=0}^{N-1} \theta_k \alpha^{n-k} u[n-k],\ \theta_k \in \mathcal{R} \text{ for } k = 0, \cdots, N-1\}. \tag{3.16}$$

It is easy to check that $\mathbf{e}_n^+$, $\mathbf{e}_n^- \in A$ for $n = 0, \cdots, N-2$ and $\mathbf{e}_{N-1}^+$, $\mathbf{e}_{N-1}^- \notin A$. But $A \bigcap \Phi(\mathbf{e}_{N-1}^+, \rho)$ is not empty since if $\theta_0 = \theta_1 = \cdots = \theta_{N-2} = 0$, $\theta_{N-1} = 1$, then $\{s_n\} \in A$ and $\| \mathbf{s} - \mathbf{e}_1^+ \| = \rho$. Therefore, if $\theta_0 = \theta_1 = \cdots = \theta_{N-2} = 0$, $\theta_{N-1} = 1$ is used as the initial setting of the equalizer, the steepest descent line (s.d.l) will never across the boundary of the cone S_{N-1}^+ because of (3.13). Hence, the equalizer with this initial setting will converge to some minimum in S_{N-1}^+, which is obviously not the global minimum $\mathbf{e}_{N-1}^+$ in this cone for it is not in $A \bigcap S_{N-1}^+$. Hence, we have proved the following theorem.

Theorem 3.1 *If the cost function of a blind equalizer satisfies conditions C1 and C2, the undesirable stable minima exist for finite length equalizer.*

Remarks:

1. In the proof of the theorem, we did not mention the distribution of the channel input symbol. Hence, the theorem can be used for any finite length equalizer with input symbols of any non-Gaussian distribution which makes the cost function satisfy condition C1 and C2. If the distribution of the input is white Gaussian, then blind equalization is not achievable. Clearly, condition C1 is not satisfied for Gaussian input.

2. The above argument only proves the local convergence of blind equalizers. If the equalizer is initialized to have $\theta_0 = 1,\ \theta_1 = 0$, then the equalizer will converge to some minimum point inside $\Phi(\mathbf{e}_0^+,\ \rho)$. However, the analysis does not indicate whether $\mathbf{e}_0^+$ is a unique minimum inside $\Phi(\mathbf{e}_0^+,\ \rho)$ or not.

3. Since the conditions C1 and C2 are satisfies by almost all known blind equalization algorithm. From Theorem 3.1, no blind equalization algorithm can get rid of the local minima caused by the finite-length effect. Such local minima are thus called *unavoidable local minima.*

4 Convergence of the SCA and the GA

The convergence performance of the GA and the SWA has been presented in [3, 4, 12] for noiseless channels. Here we will first extend these analysis results to SCA. Another objective is to study the global convergence of SCA and the GA under white Gaussian channel noise.

A. Convergence of SCA in noiseless environments

Similar to Godard equalizer and the Shalvi-Weinstein equalizer [12], the finite-length equalizer using the SCA have the following convergence properties.

Theorem 4.1 *For (finite-length or infinite-length) standardized cumulant equalizers, suppose the initial equalizer parameter setting causes the initial overall parameter vector to satisfy* $\mathbf{s}^{in} \in A \bigcap S_n^{\pm}$. *If the initial equalizer output* y_n *satisfies*

$$\frac{|K_{y_n}^{2m}|}{|K_{a_n}^{2m}|} > 0.5, \tag{4.1}$$

where

$$K_{y_n}^{2m} = \frac{C_{y_n}^{2m}}{(C_{y_n}^2)^m},\quad K_{a_n}^{2m} = \frac{C_{a_n}^{2m}}{(C_{a_n}^2)^m}, \tag{4.2}$$

then under very small minimization step size, the equalizer will cause $\mathbf{s}$ *to converge to a minimum point inside* $A \bigcap S_n^{\pm}$.

Theorem 4.2 *Let A be the attainable set of a finite-length standardized cumulant equalizer. Then*

1. *If $\mathbf{e}_n^{\pm} \in A \bigcap S_n^{\pm}$, then there is only one minimum point $\mathbf{e}_n^{\pm}$ in $A \bigcap S_n^{\pm}$, and there is no minimum point on the boundary $A \bigcap B_n^{\pm}$.*

2. *If $\mathbf{e}_n^{\pm}$ is near but not in $A \bigcap S_n^{\pm}$ (not attainable), then there must exist only one minimum point in $A \bigcap S_n^{\pm}$ near $\mathbf{e}_n^{\pm}$ while all other possible minima are near the boundary $A \bigcap B_n^{\pm}$.*

The proofs of the above two theorems are similar to the proofs of Theorem 5.1 and Theorem 5.2 in [12], respectively. they are thus omitted here. With these theorems, the initialization strategy for GA and SWA discussed in [12] can also be used for SCA.

If the length of a SCA equalizer is double-infinite (an impractical abstraction), then $\mathbf{e}_n^{\pm}$ for all n are in the attainable set A. From Theorem 4.2, the SCA equalizer will converge to some $\mathbf{e}_n^{+}$ or $\mathbf{e}_n^{-}$. Hence, SCA has no inherent local minima.

B. Global convergence of the GA and the SCA under Gaussian channel noise

We will first study the convergence of the Godard algorithm under white Gaussian noise. We will assume that the signal-to-noise ratio is very high.

From the definition of (2.12), the Godard cost function can be written as

$$g(\theta) = f(\mathbf{s}) = \frac{1}{4} m_2^2 \{-(3-r)\sum_n s_n^4 + 3(\sum_n s_n^2)^2 - 2r\sum_n s_n^2 + r^2 \tag{4.3}$$

$$+ 6\eta \sum_n s_n^2 \sum_n \theta_n^2 + 3\eta^2 (\sum_n \theta_n^2)^2 - 2r\eta \sum_n \theta_n^2\}, \tag{4.4}$$

where

$$r = \frac{m_4}{m_2^2}, \qquad \eta = \frac{\sigma^2}{m_2}. \tag{4.5}$$

From the above equation, the global minima of the Godard equalizer will not be $\mathbf{e}_n^{\pm}$. However, if the signal-to-noise ratio is high, the global minima will be near $\mathbf{e}_n^{\pm}$. Then the global minima can be expressed as $\mathbf{e}_n^{\pm} + \Delta\mathbf{s}$. The parameters of the equalization can be written as $\check{\mathbf{h}} + \Delta\theta$, where $\{\check{h}_n\} = \mathcal{Z}^{-1}\{H^{-1}(z)\}$, and $\Delta s_n = \sum_k h_k \Delta\theta_{n-k}$. Using Taylor expansion, we have

$$\begin{aligned} f(\mathbf{s}) = \quad & f(\mathbf{e}_0^{+}) + \tfrac{1}{2} m_2^2 [(3-r)\textstyle\sum_n (\Delta s_n)^2 + 3(r-1)(\Delta s_0)^2 + O(\| \Delta \mathbf{s} \|^2)] \\ & + m_2^2 \eta [(3-r)\textstyle\sum_n t_n \Delta\theta_n + 3d_2 \sum_n h_{-n}\Delta\theta_n) + O(\| \Delta\theta \|)], \end{aligned} \tag{4.6}$$

where

$$d_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|H(\omega)|^m} d\omega. \tag{4.7}$$

Ignoring the $O(\| \Delta \mathbf{s} \|^2)]$ and $\eta O(\| \Delta\theta \|)]$ terms in the above expression, a direct calculation yields that the minimum of the cost function will satisfy

$$(3-r)\Delta S(\omega)H^*(\omega) + 3(r-1)\Delta s_0 H^*(\omega) + (3-r)\eta H^{-1}(\omega) + 3\eta d_2 H^*(\omega) \approx 0 \quad (4.8)$$

where $\Delta S(\omega)$ is the Fourier transform of $\{\Delta s_n\}$, and $*$ denotes complex conjugate. Hence,

$$\Delta S(\omega) \approx -(\frac{1}{|H(\omega)|^2} + \frac{3(2-r)}{2r}d_2)\eta. \quad (4.9)$$

The transfer functions of the equalized system and the equalizer are

$$S(\omega) \approx 1 - (\frac{1}{|H(\omega)|^2} + \frac{3(2-r)}{2r}d_2)\eta, \quad (4.10)$$

and

$$\Theta(\omega) \approx \frac{S(\omega)}{H(\omega)}. \quad (4.11)$$

The mean square error (MSE) after equalization will be

$$(MSE)_{GA} \approx d_2\eta - (d_4 - \frac{9(2-r)^2}{4r^2}d_2^2)\eta^2. \quad (4.12)$$

Similar derivation yields that the transfer function of the equalized system and the equalizer for SCA is

$$S(\omega) \approx 1 - (\frac{1}{|H(\omega)|^2} - \frac{1}{2}d_2)\eta, \quad (4.13)$$

and

$$\Theta(\omega) \approx \frac{S(\omega)}{H(\omega)}, \quad (4.14)$$

and the MSE is

$$(MSE)_{SCA} \approx d_2\eta - (d_4 - \frac{1}{4}d_2^2)\eta^2. \quad (4.15)$$

If the channel inverse and the Wiener filter are used as equalizers, the MSE's are

$$(MSE)_{inv} = d_2\eta, \quad (4.16)$$

and

$$(MSE)_{Win} \approx d_2\eta - d_4\eta^2 \quad (4.17)$$

respectively. From (4.15), $(MSE)_{SCA}$ is between $(MSE)_{inv}$ and $(MSE)_{SCA}$. However, if the signal of a PAM system is over 2 levels, then $1.68 \le r \le 1.8$, so $0 < d_4 - \frac{9(2-r)^2}{4r^2}d_2^2 < d_4$. Thus $(MSE)_{GA}$ is also between $(MSE)_{inv}$ and $(MSE)_{SCA}$.

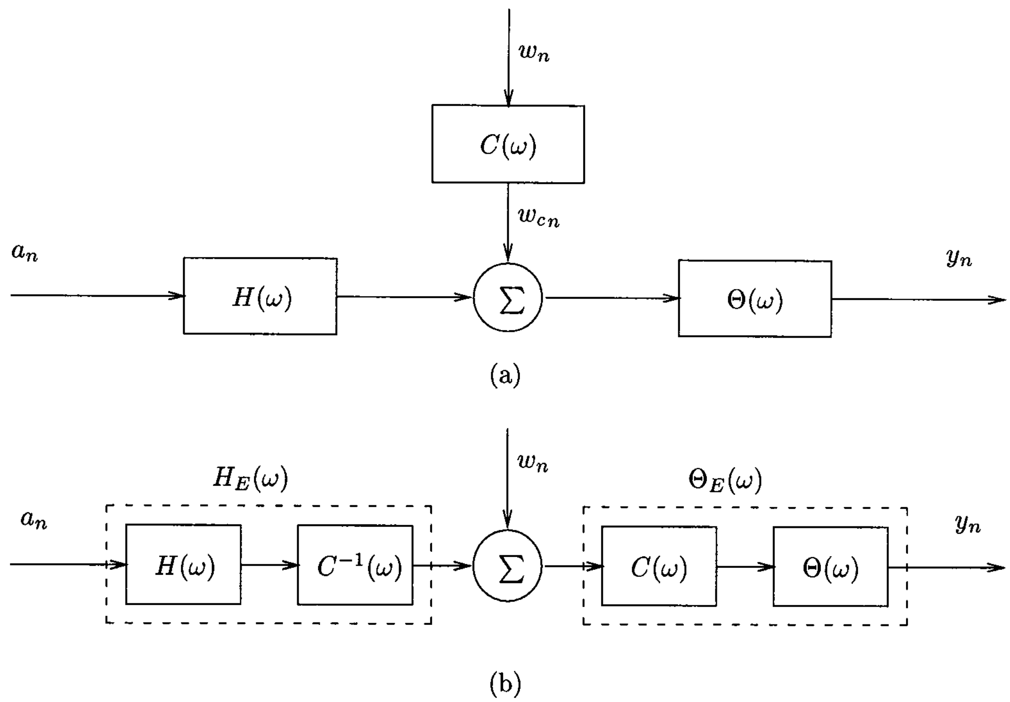

Figure 3: (a) System with colored channel noise, and (b) its equivalent white noise model.

When the channel noise w_{cn} is a regular, colored Gaussian process with power spectrum $W(\omega)$, there exists a minimum-phase function $C(\omega)$ such that

$$W(\omega) = C(\omega)C^*(\omega). \tag{4.18}$$

Therefore, the colored Gaussian noise w_{cn} is equivalent to the output of a system with transfer function $C(\omega)$ driven by white Gaussian input w_n. The system with regular colored channel noise can be demonstrated by Figure 4(a), which is equivalent to Figure 4(b). In Figure 4(b),

$$H_E(\omega) = \frac{H(\omega)}{C(\omega)}, \tag{4.19}$$

and

$$\Theta_E(\omega) = \Theta(\omega)C(\omega). \tag{4.20}$$

Hence, the above analysis results can be directly used here.

5 Inherent Local Minima of Decision-Directed Equalizers

For the blind equalization algorithms defined by the prediction error, the convergence analysis is not as easy as that of GA or SCA since the general analytical expression of the cost function can not be obtained. The local convergence of decision-directed equalizer (DDE) implemented with FIR filter is demonstrated in [11, 14, 15], which reveals the unavoidable local convergence of DFE. We will illustrate the local convergence of the decision-directed equalizer (DFE) and the Stop-and-Go algorithm (SGA) by specifying their cost functions in some operating regions.

A noiseless PAM communication system with a double-infinite-length equalizer is considered here. Then the attainable set of the equalizer $A = \ell^1(\mathcal{R})$. Assume the PAM channel input is i.i.d., and is uniformly distributed over the set $\{-3,\ -1,\ 1,\ 3\}$. Double infinite length SGA equalizer with $\beta = 2$ is used in this system. We will show that $\{s_n\} = \{\frac{13}{31}(\delta[n-1]+\delta[n]+\delta[n+1])\}$ is one of the inherent local minimum of the SGA. To show this, we first find its cost function near $\mathbf{s}_o = \{\frac{13}{31}(\delta[n-1]+\delta[n]+\delta[n+1])\}$.

From (2.16), (2.17) and (2.18), we can find that

$$\Psi(y) = \begin{cases} \frac{1}{2}(|y|-1)^2 & |y| < 1 \\ 0 & 1 \leq |y| \leq 3 \\ \frac{1}{2}(|y|-3)^2 & |y| > 3 \end{cases} \tag{5.1}$$

From (2.8), for any $\mathbf{s}$ near $\mathbf{s}_o$, we have

$$\begin{array}{ll} |y_n| < 1 & \text{if } (a_{n+1},\ a_n,\ a_{n-1}) = \pm(3,-1,-1), \pm(-1,3,-1), \pm(-1,-1,3) \\ & \qquad\qquad\qquad\quad \pm(1,1,-1), \pm(1,-1,1), \pm(-1,1,1) \\ |y_n| > 3 & \text{if } (a_{n+1},\ a_n,\ a_{n-1}) = \pm(3,\ 3,\ 3) \\ 1 < |y_n| < 3 & \text{otherwise} \end{array} \tag{5.2}$$

Therefore, by direct calculation, we can get the cost function near $\mathbf{s}_o$

$$f(\mathbf{s}) = \frac{65}{32} \sum_{n \neq -1,\ 0,\ 1} s_n^2 + \frac{1}{32} \sum_{n=-1,\ 0,1} (31 s_n^2 - 26 s_n + 7). \tag{5.3}$$

From (5.3), it is obvious that $\mathbf{s}_o$ is an inherent local minimum of the Stop-and-Go algorithm. Since the cost function is symmetric on s_n, $\{\pm\frac{13}{31}(\delta[n-n_1] \pm \delta[n-n_2] \pm \delta[n-n_3])\}$ are inherent local minima for all different $n_1,\ n_2$, and n_3 .

If the decision feedback equalizer is used in the above communication system, by the similar derivation, it can be shown that $\{\pm\frac{37}{80}(\delta[n-n_1] \pm \delta[n-n_2] \pm \delta[n-n_3])\}$ are inherent local minima for any different integers $n_1,\ n_2$, and n_3 .

From the above discussion, both the Stop-and-Go algorithm and the decision feedback equalizers may have ill-convergence even if the equalizers are double infinite. Hence, the center-tap initialization strategy [7, 12] can not avoid the local convergence. Such behavior will be demonstrated by computer simulations in the next section.

6 Computer simulation results

In this section, we will test the global convergence of the Godard algorithm under Gaussian noise and the local convergence of the Stop-and-Go algorithm.

A. Global convergence of the Godard equalizer under noise

The noisy channel considered in this example is given by

$$H(\omega) = \frac{e^{-j\omega} - 0.8}{1 - 0.8e^{-j\omega}}. \tag{6.1}$$

The channel noise is white Gaussian. A 100-tap Godard equalizer with center-tap initialization is used in the system.

When the channel input is binary, from (4.12), the MSE of the equalized system will be $(1 + 1.25\eta)\eta$. The simulated result is in Figure 3. If the channel input is 4-level, the MSE of the equalized system will be $(1 - 0.9184\eta)\eta$. The simulated result is in Figure 4.

From Figure 3 and Figure 4, our analysis fits the simulation results well.

B. Local convergence of the Stop-and-Go equalizer

Noiseless channel in PAM communication system considered in this example is

$$h(n) = \frac{13}{31}(\delta[n] + \delta[n-1] + \delta[n-3]). \tag{6.2}$$

The channel input a_n is i.i.d., and uniform over the set $\{-3,\ -1,\ 1,\ 3\}$. A 100-tap Stop-and-Go equalizer with $\beta = 2$ and small step size $\mu = 0.0002$ is used. Intersymbol interference is used to measure the performance of the equalized system, which is defined as

$$ISI = \frac{\sum_n s_n^2 - \max_n s_n^2}{\max_n s_n^2}. \tag{6.3}$$

If the central-tap initial value, i.e. $\theta_0 = 1$ and $\theta_n = 0$ for all $n \neq 0$, is used, the ISI and the impulse response of the equalized system are shown in Figure 5. From the figure, the equalizer can not remove the ISI and it gets stuck on an inherent local minimum. However, if the initial value of is set such that $g_0 = g_1 = 1$ and the rest of the coefficients are zero, the equalizer is able to remove the ISI as is shown by Figure 6.

7 Conclusion

This paper is devoted to the convergence analysis of blind equalization algorithms. We have shown that every blind equalization algorithm implemented with FIR filters possesses some *unavoidable local minima.* Since unavoidable local minima are common to all blind adaptive equalizers, their regions of attractions, instead of their existence, should be used to judge the performance of an algorithm. The local convergence property of blind equalization algorithms under the double-infinite equalizer abstraction is different from algorithm to algorithm and is inherent to the cost function selection. From the previous work in [3, 4, 5, 7, 12] and the discussion of this paper, it is known that that BGR algorithm [1], the Sato algorithm (SA) [18], the decision-directed equalizer (DDE) [8], and the Stop-and-Go algorithm (SGA) [17] all have inherent local minima. On the other hand, the Godard algorithm (GA) [10], the Shalvi-Weinstein algorithm (SWA) [24], and standard cumulant algorithms (SCA) [6] have no inherent local minimum. We also show that under high SNR, the Godard algorithm, the Shalvi-Weinstein algorithm, and standard cumulant algorithms will not amplify white channel noise. Thus, GA, SWA, and SCA should be favored adaptive blind equalization algorithms. Our analysis results are confirmed by computer simulations.

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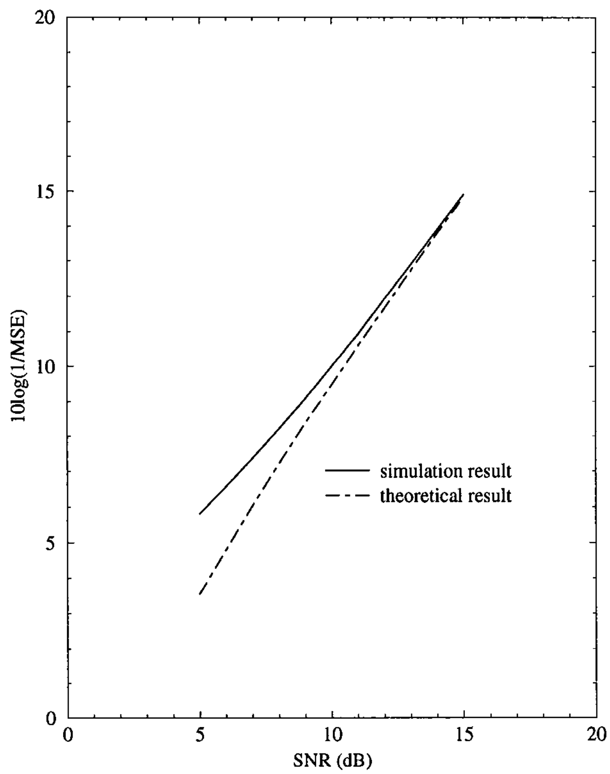


Figure 4: The MSE of the equalizer output when the channel input is binary

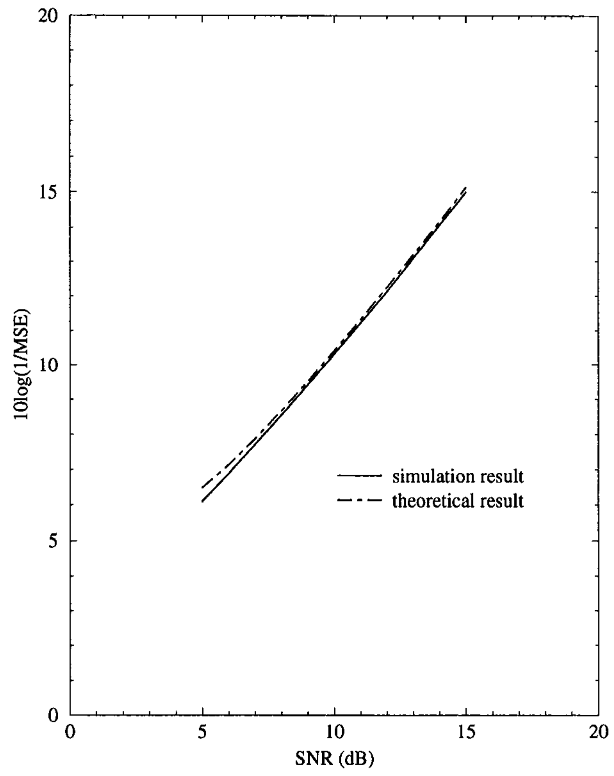


Figure 5: The MSE of the equalizer output when the channel input is of 4-level

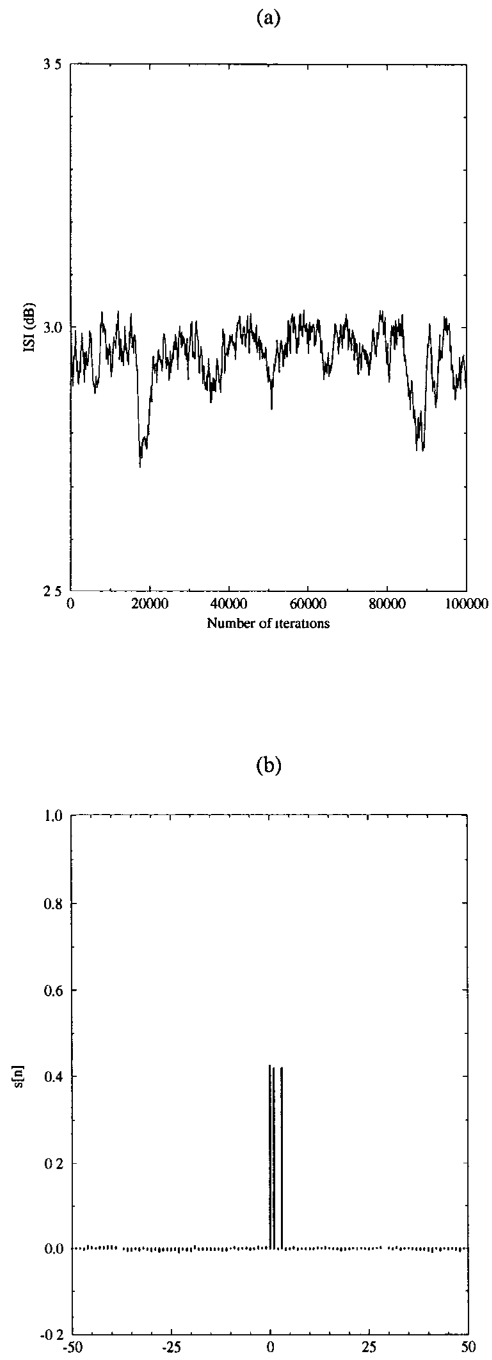
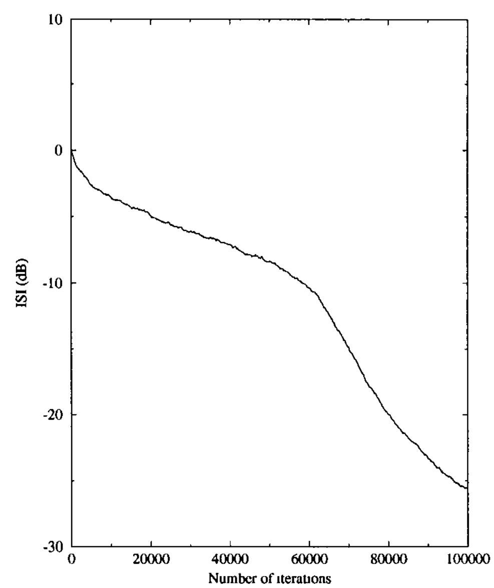


Figure 6: The ISI (a) and the impulse response (b) of the equalized system when the initial values of g_0 is unit, and the rest of g_n are zero.

(a)



(b)

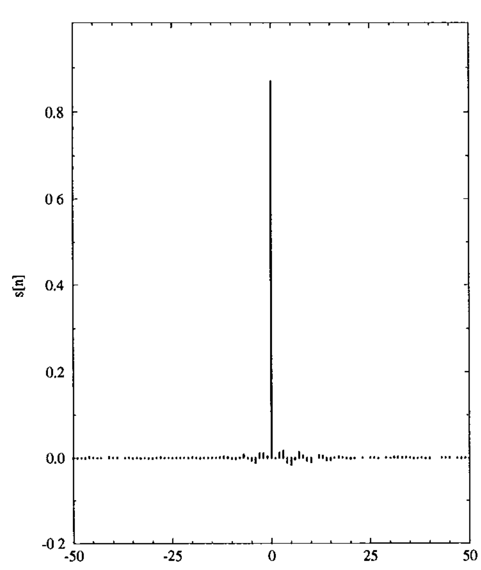


Figure 7: The ISI (a) and the impulse response (b) of the equalized system when the initial values of g_0 and g_1 are unit, and the rest of g_n are zero.