Nuclear Reactions
Some Basics

II. Reaction Kinematics

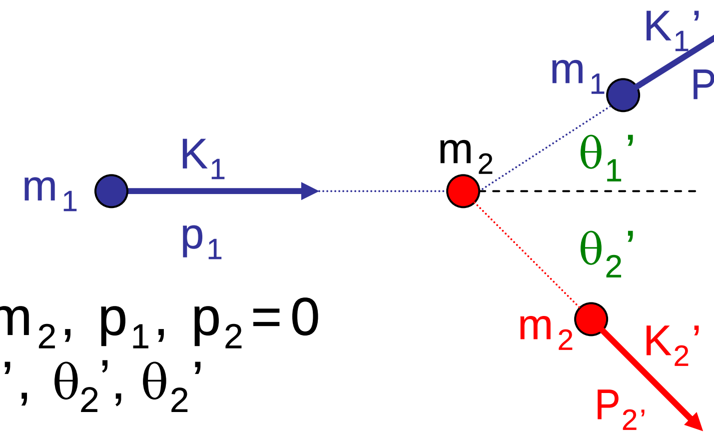
Reaction Kinematics

- A beam incident on a target will result in many different reactions, including elastic and inelastic scattering, representing different collisions of beam particles with different nuclei.
- To isolate a specific reaction (to measure, e.g., its cross section or some other prediction of theory), in addition to identifying the correct particles involved, we need to “look” at the right place, i.e. isolate products moving in the “correct” momentum (i.e. energy and direction).
- To assist in identifying the correct energy and direction of emitted particles, we utilize the **kinematics** of the reaction, i.e. the predictions of:
  - Conservation of energy
  - Conservation of linear momentum

  for this reaction (for CEBAF energies, must use relativistic expressions)
- Two-body kinematics
  - Used when two particles are present in the “final state” (i.e. after collision)
- Three-body kinematics
  - Used when three particles are present in the final state

Two-Body Kinematics

Elastic Scattering in a plane (non-relativistic)



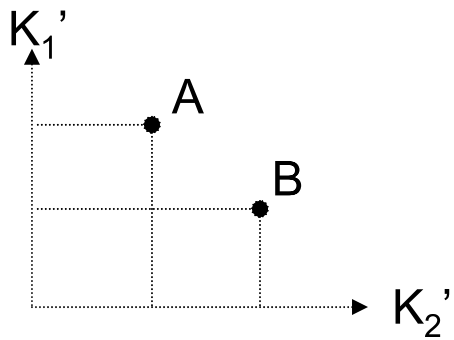
Given initial state: m_1, m_2, p_1, $p_2 = 0$
Find final state: K_1', K_2', θ_2', θ_2'

- Recall: $K = \frac{1}{2} mv^2$, $p = mv$, $K = p^2/2m$
- Only relevant physics laws are conservation of energy and momentum
- Conservation of Energy: $K_1 = K_1' + K_2'$ (1)
- Conservation of Momentum: $p_1 = p_1'\cos\theta_1' + p_2'\cos\theta_2'$ (2)
  $0 = p_1'\sin\theta_1' - p_2'\sin\theta_2'$ (3)
- We have 3 equations, and 4 unknowns (K_1', K_2', θ_1', θ_2')
- Therefore K's and θ's are not uniquely determined
- If given one of the final state variables we can calculate the other three

Two-Body Kinematics
Elastic Scattering in a plane (non-relativistic)

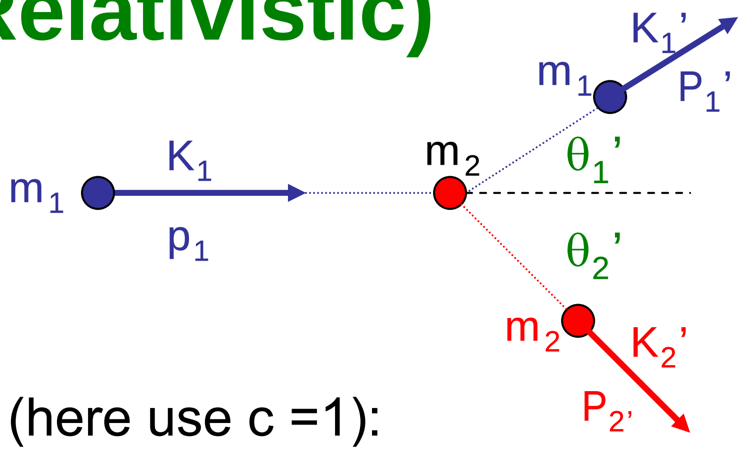
Some Observations

- If the initial conditions (m_1, m_2, $\vec{p}_1$, $\vec{p}_2$=0) are known in an actual elastic scattering experiment, and if a final state particle is chosen to be detected at an angle θ_1, two-body kinematics determine a unique angle θ_2 at which the other particle will emerge as well as both kinetic energies.
- Said differently, if two detectors are to detect particles corresponding to a single elastic scattering event, they have to be placed at "conjugate" angles, i.e. angles related through two-body kinematics
- Note: A **single** elastic scattering **event** (labeled A) detected at conjugate angles $(\theta_1, \theta_2)_A$ is represented by a **single point** on a plot of K_1' vs. K_2' . For the same initial conditions, another event with particles detected at a different set of conjugate angles $(\theta_1, \theta_2)_B$ will produce a different combination of K_1' and K_2' and will be represented by a different single point (labeled B) on the same plot.



Two-Body Kinematics
Elastic Scattering (Relativistic)

Given: m_1, m_2, K_1, θ_1'
Find: K_1', K_2', θ_2'



- Conservation of (total) Energy (here use c =1):

$$E_1 + m_2 = E_1' + E_2' \qquad (1)$$

- Conservation of Momentum:

$$p_1 = p_1'\cos\theta_1' + p_2'\cos\theta_2' \qquad (2)$$

$$0 = p_1'\sin\theta_1' - p_2'\sin\theta_2' \qquad (3)$$

- Recall: $E_1 = K_1 + m_1$ and $E_1^2 = p_1^2 + m_1^2$

  $E_1' = K_1' + m_1$ and $E_1'^2 = p_1'^2 + m_1^2$

  $E_2' = K_2' + m_2$ and $E_2'^2 = p_2'^2 + m_2^2$

- Solve (1), (2), (3) for K_1', K_2', θ_2'

Relativistic Two-Body Kinematics Elastic Scattering – Results (1)

$$E_1' = \frac{A_1(E_1 + m_2) \pm p_1 \cos\theta_1' \sqrt{A_1^2 - 4m_1^2\left[(E_1 + m_2)^2 - p_1^2\cos^2\theta_1'\right]}}{2\left[(E_1 + m_2)^2 - p_1^2\cos^2\theta_1'\right]}$$

$$E_2' = \frac{A_2(E_1 + m_2) \pm p_1 \cos\theta_2' \sqrt{A_2^2 - 4m_2^2\left[(E_1 + m_2)^2 - p_1^2\cos^2\theta_2'\right]}}{2\left[(E_1 + m_2)^2 - p_1^2\cos^2\theta_2'\right]}$$

$$\cot\theta_2' = \frac{-(1 + \rho_1\rho_2)\cot\theta_1' \pm (\rho_1 + \rho_2)\sqrt{\gamma^2(1 - \rho_1^2) + \cot^2\theta_1'}}{1 - \rho_1^2}$$

Where A_1, A_2, ρ_1, ρ_2, γ are given by:

Relativistic Two-Body Kinematics Elastic Scattering – Results (2)

$$A_1 = 2\left(m_1^2 + m_1 m_2 + m_2 K_1\right)$$

$$A_2 = 2\left(m_2^2 + m_1 m_2 + m_2 K_1\right)$$

$$\rho_1 = \frac{A_1 p_1}{\left(E_1 + m_2\right)\sqrt{A_1^2 - 4m_1^2\left[\left(E_1 + m_2\right)^2 - p_1^2\right]}}$$

$$\rho_2 = \frac{A_2 p_1}{\left(E_1 + m_2\right)\sqrt{A_2^2 - 4m_2^2\left[\left(E_1 + m_2\right)^2 - p_1^2\right]}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v_{cm}^2}{c^2}}} = \frac{E_1 + m_2}{\sqrt{m_1^2 + m_2\left(2E_1 + m_2\right)}}$$

Interactive program that calculate 2-body kinematics:
http://www.calstatela.edu/academic/nuclear_physics/2bdkin.html

Relativistic Two-Body Kinematics Elastic Scattering - Observations

- **Note:** Not any θ_1, θ_2 pairs are possible in two-body kinematics
  - Square root in E_1, E_2 must be > 0

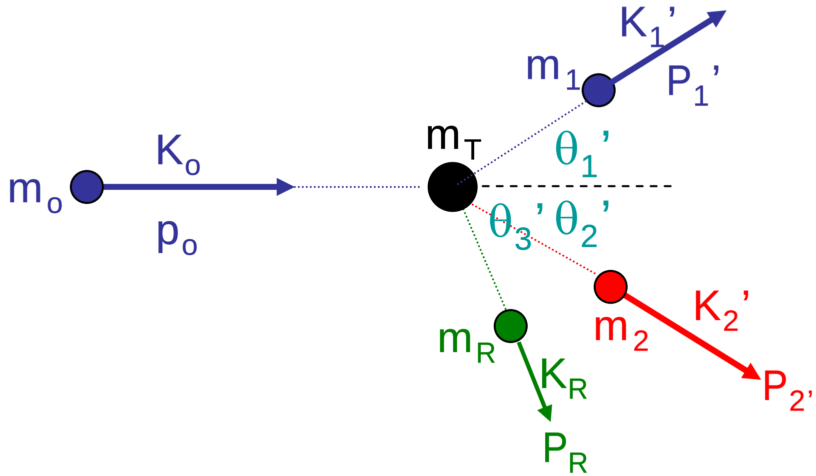
$$\sin^2\theta_1' \leq \frac{A_1^2 - 4m_1^2\left[(m_1+m_2)^2 + 2m_2K_1\right]}{4m_1^2p_1^2}$$

$$\sin^2\theta_2' \leq \frac{A_2^2 - 4m_2^2\left[(m_1+m_2)^2 + 2m_2K_1\right]}{4m_2^2p_1^2}$$

- **Note:** If $m_1 = m_2$ then $\theta_1' + \theta_2' < \pi/2$ (unlike classical mechanics)

- **Interactive program** that calculate 2-body kinematics:
  - http://www.calstatela.edu/academic/nuclear_physics/2bdkin.html

Three-Body Kinematics
Nuclear Reaction (Relativistic)



- Conservation of energy and momentum hold again.

$$E_o + m_T = E_1' + E_2' + E_R' \qquad (1)$$

$$p_o = p_1'\cos\theta_1' + p_2'\cos\theta_2' + p_R\cos\theta_3' \qquad (2)$$

$$0 = p_1'\sin\theta_1' - p_2'\sin\theta_2' + p_R\sin\theta_3' \qquad (3)$$

  – In addition, but ignore for now, there may be "out-of-plane" (φ) momentum components

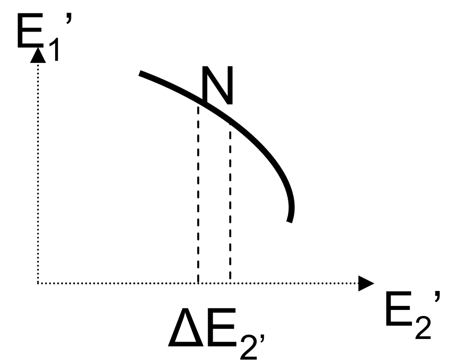
- Here we have **3** equations and **6** unknowns (E_1', E_2', E_R', θ_1', θ_2', θ_3')
- Need to know **3** variables in the final state to compute everything else

Computing Relativistic Three-Body Kinematics

- Will not solve analytically expressions for unknowns here.
- Go to: http://www.calstatela.edu/academic/nuclear_physics/kin3b.htm
  - On-line program that computes relativistic three-body kinematics

Relativistic Three-Body Kinematics Nuclear Reaction – An Observation

- If 2 of the 3 final state unknowns are specified (e.g. by specifying the energy and direction of one of the detected particles) the reaction is represented on the E_1' vs E_2' plot for the remaining 2 particles by a **curve** (compared to a point in 2-body scattering)



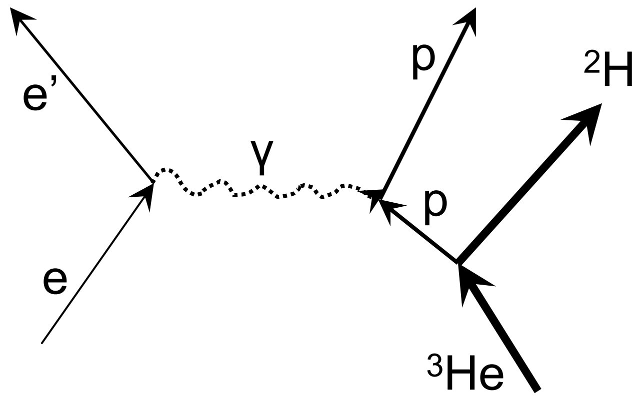
- In computing the 5-fold cross section:

$$\frac{d^5\sigma}{d\Omega_{1'} d\Omega_{2'} dE_{2'}} = \frac{Ne}{Qnx\Delta\Omega_{1'}\Delta\Omega_{2'}\Delta E_{2'}}$$

N corresponds to events on the curve projected to "byte" $\Delta E_{2'}$

The $^3He(e,e'p)^2H$ Reaction Mechanism

- The presumed reaction mechanism is represented by the diagram:



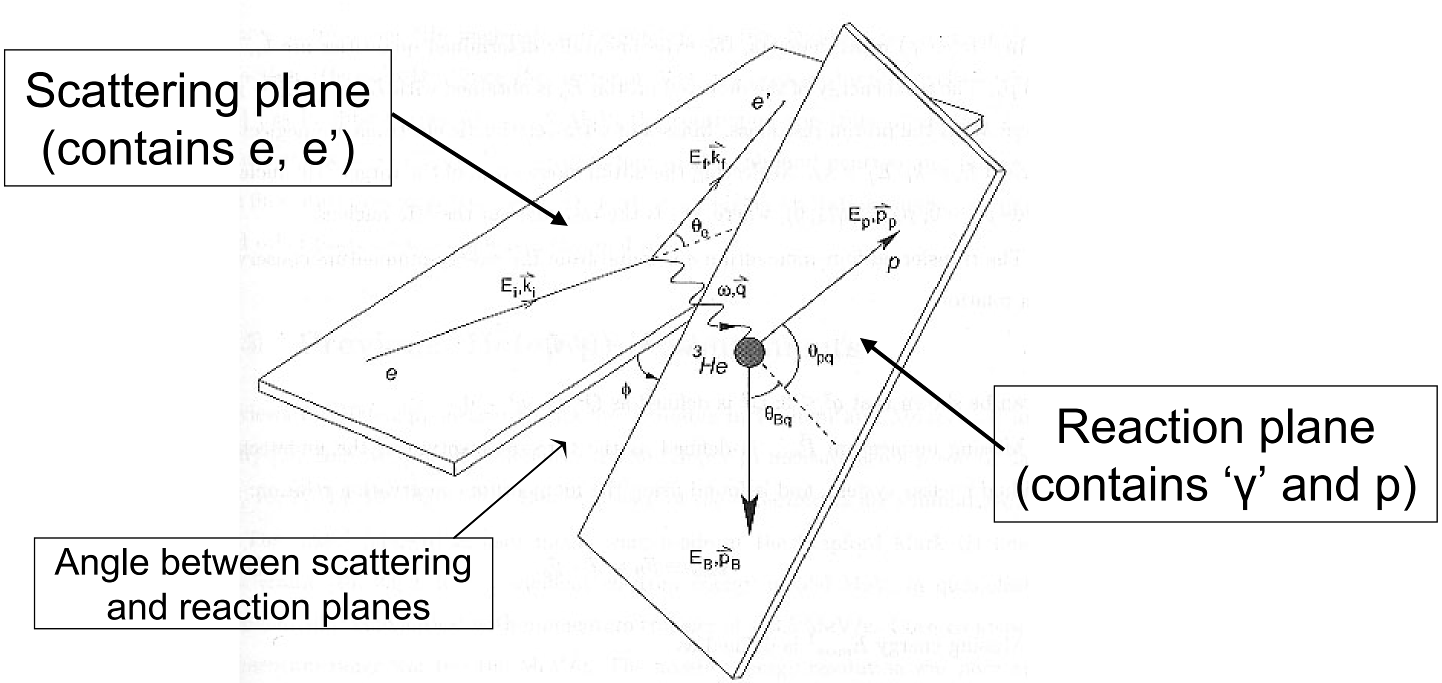
- This is known as the Plane Wave Impulse Approximation, i.e. virtual photon interacts only with a single bound proton, which is subsequently ejected from the target and detected, while leaving the rest of ^3He (i.e. ^2H) unaffected (spectator)
- Valid at high energies (e.g. CEBAF)
- Note: Only one virtual photon is assumed to be exchanged

Definitions of Kinematic Variables

- Incident electron: momentum $\mathbf{k_i}$, total energy E_i , mass m_e
- Scattered (and detected) electron: $\mathbf{k_f}$, E_f, m_e
- Ejected (and detected) proton: $\mathbf{p_p}$, E_p, m_p
- Target nucleus (^3He): p_A, E_A, M_A
- Left over nucleus (^2H, spectator): $\mathbf{p_B}$, E_B.
- NOTES (all quantities in lab frame):
  - Target momentum $p_A=0$ (target at rest in lab)
  - Target total energy $E_A = m_A c^2$ (target at rest, rest energy only)
  - $E_i = T_i$ (kinetic energy of incident electron) + $m_e c^2$ (its rest energy)
    - Here ignore $m_e c^2 = 0.511$ MeV since T_e =1-6 GeV at CEBAF >> $m_e c^2$

    Therefore: $E_i \approx T_i \approx k_i$ (recall $E^2 = k^2c^2 + (m_o c^2)^2$, E in MeV/$c^2$, k in Mev/c)
  - $E_f \approx T_f \approx k_f$ (similarly)
  - $E_p = \sqrt{p_p^2 c^2 + (M_p c^2)^2}$ (here can't neglect rest energy, heavier proton is not ultra-relativistic)

Some Additional Kinematic Variables



- Incident electron scattering angle: θ_e
- Angle between scattering and reaction planes: φ
  - "In-plane" kinematics: $\varphi = 0$ (p "forward" of **q**), $\varphi = \pi$ (p "backward" of **q**)
  - "out-of-plane" kinematics: $\varphi \neq 0$
- Angle between q and p_p: θ_{pq}

Additional Kinematic Variables

- In a typical (e,e'p) experiment we measure:
  - E_i, $\mathbf{k_i}$ (incident beam energy, momentum)
    - Recall: $E_i = T_i = k_i$ (ultra-relativistic electrons)
  - E_f, $\mathbf{k_f}$ (scattered electron energy, momentum, using Hall A HRS)
    - Similarly recall: $E_f = T_f = k_f$ (ultra-relativistic electrons)
  - $\mathbf{p_p}$ (momentum of ejected proton (using Hall A HRS)
- Virtual photon momentum **q** may be calculated from conservation of momentum at the e-e'-'γ vertex ($\mathbf{k_i} = \mathbf{k_f} + \mathbf{q}$):

$$\mathbf{q} = \mathbf{k_i} - \mathbf{k_f}$$

- Virtual photon energy ω may be calculated from conservation of energy at the same vertex ($E_i = E_f + \omega$):

$$\omega = E_i - E_f$$

- **q** and ω are also the momentum and energy **transferred** to (and absorbed by) the target nucleus by the virtual photon through this collision.

Missing Momentum and Energy

- **"Missing momentum"**, $\mathbf{p}_{\mathbf{miss}}$, is the momentum of the undetected recoil nucleus (^2H). It may be calculated from conservation of momentum at the 'γ'-^3He-p vertex ($\mathbf{q} = \mathbf{p}_p + \mathbf{p}_B$). Therefore:

$$\mathbf{p}_{miss} \equiv \mathbf{p}_B = \mathbf{q} - \mathbf{p}_p$$

- "**Missing energy**", E_{miss}, is the unaccounted part of the energy transfer ω after kinetic energies T_p and T_B of the known pieces of the broken up nucleus are subtracted from it. i.e.

$$E_{miss} = \omega - T_p - T_B$$

  - Note: $T_p = E_p - M_p c^2 = (p_p^2 c^2 + M_p^2 c^4)^{1/2} - M_p c^2$
    $T_B = E_B - M_B c^2 = (p_B^2 c^2 + M_B^2 c^4)^{1/2} - M_B c^2$

- Note: Since the values of k_i, k_f, p_p, are known (measured), the values of q, ω, p_{miss} and E_{miss} can be calculated **for each event** of this reaction.

The E_{miss} Spectrum for $^3He(e,e'p)X$

- E_{miss} consists of the energy required to **"separate"** (un-bind) the ejected proton from the **target** nucleus plus any **"excitation"** energy the **recoil** nucleus might have (above its lowest, ground state). i.e.

$$E_{miss} = E_{sep} + E_{exc}$$

$$\text{where: } E_{sep} = M_p c^2 + M_B c^2 - M_A c^2$$

- Since ^3He does not have any excited states, and **if** the only $e+^3He$ reaction channel possible is $^3He(e,e'p)^2H$, there should be only one peak in the E_{miss} spectrum, at the value of E_{sep}, which can be calculated to be 5.49 MeV (from $m_A = m_p + m_{2H} + E_{sep}$).
- Of course, there are additional channels available, including the 3-body breakup channel $^3He(e,e'p)pn$ and particle creation at higher E_{miss}, leading to additional features in the E_{miss} spectrum.
- The ^2H nucleus (deuteron) has a binding energy of (i.e. it breaks up at) 2.22 MeV. Therefore a second peak would be expected in the spectrum at E_{miss} = 5.49 + 2.22 = 7.71 MeV. See Figure in next slide…

The E_{miss} Spectrum for ^{3}He(e,e'p)X #1

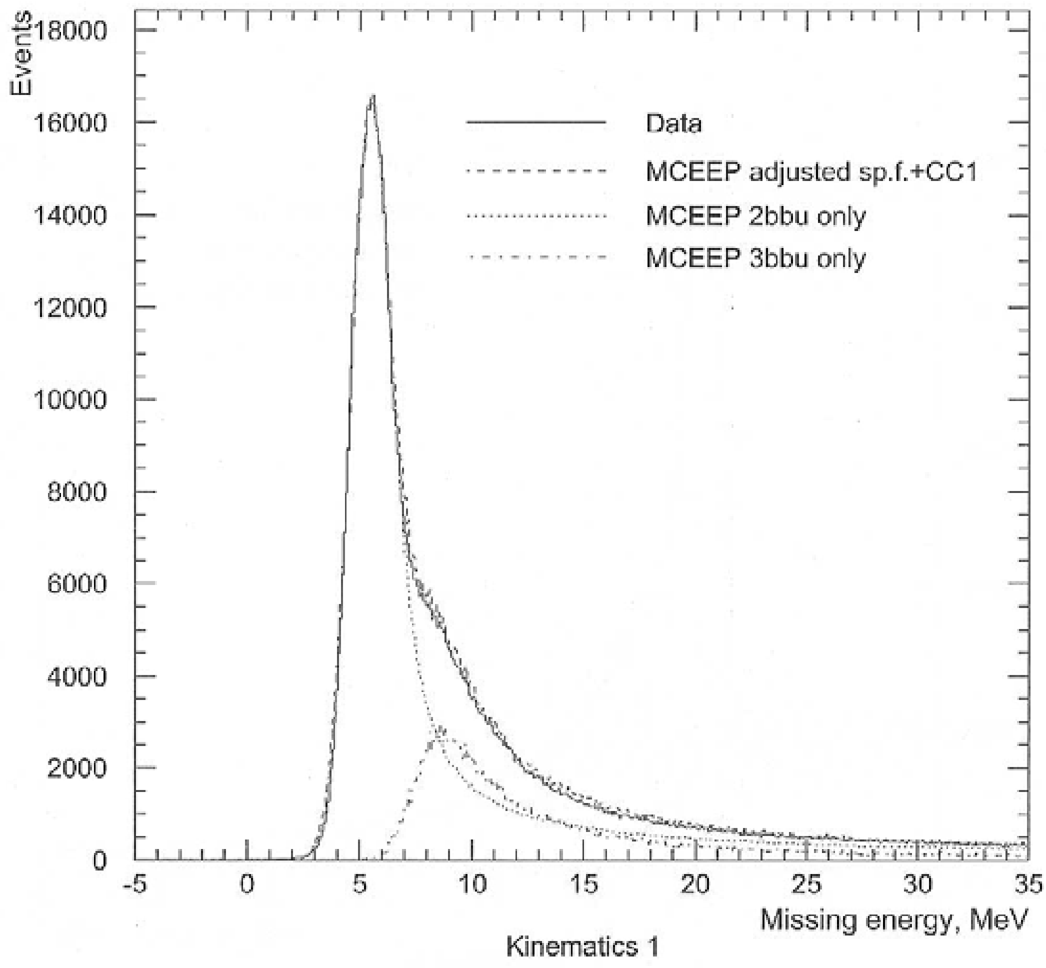


Figure 5-21: ^{3}He(e,e'p) missing energy distributions at kinematics 1, E_{beam} = 4.8 GeV, $P_{miss} = 0$.

The E_{miss} Spectrum for $^3He(e,e'p)X$ #2

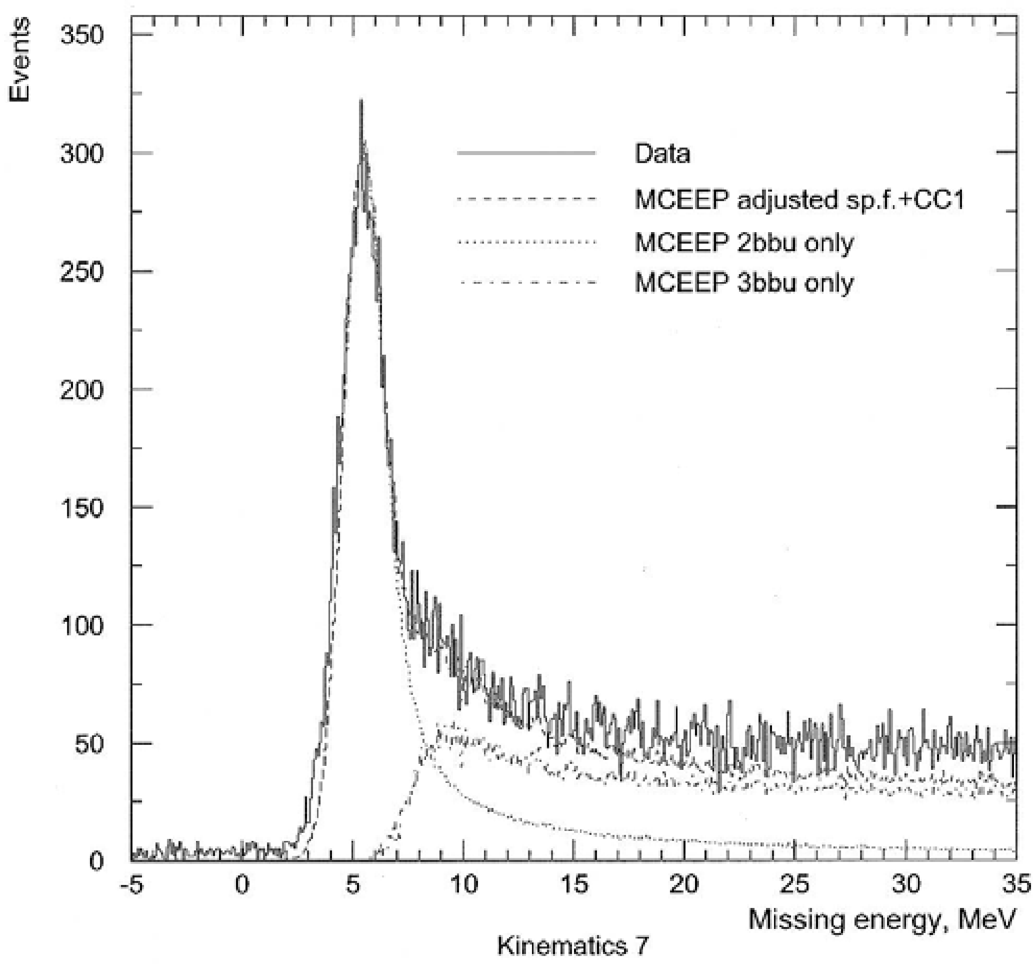


Figure 5-23: $^3He(e,e'p)$ missing energy distributions at kinematics 7, E_{beam} = 4.8 GeV, P_{miss} = 300 MeV/c, the detected proton is back of $\vec{q}$.

The E_{miss} Spectrum for $^3He(e,e'p)X$ #3

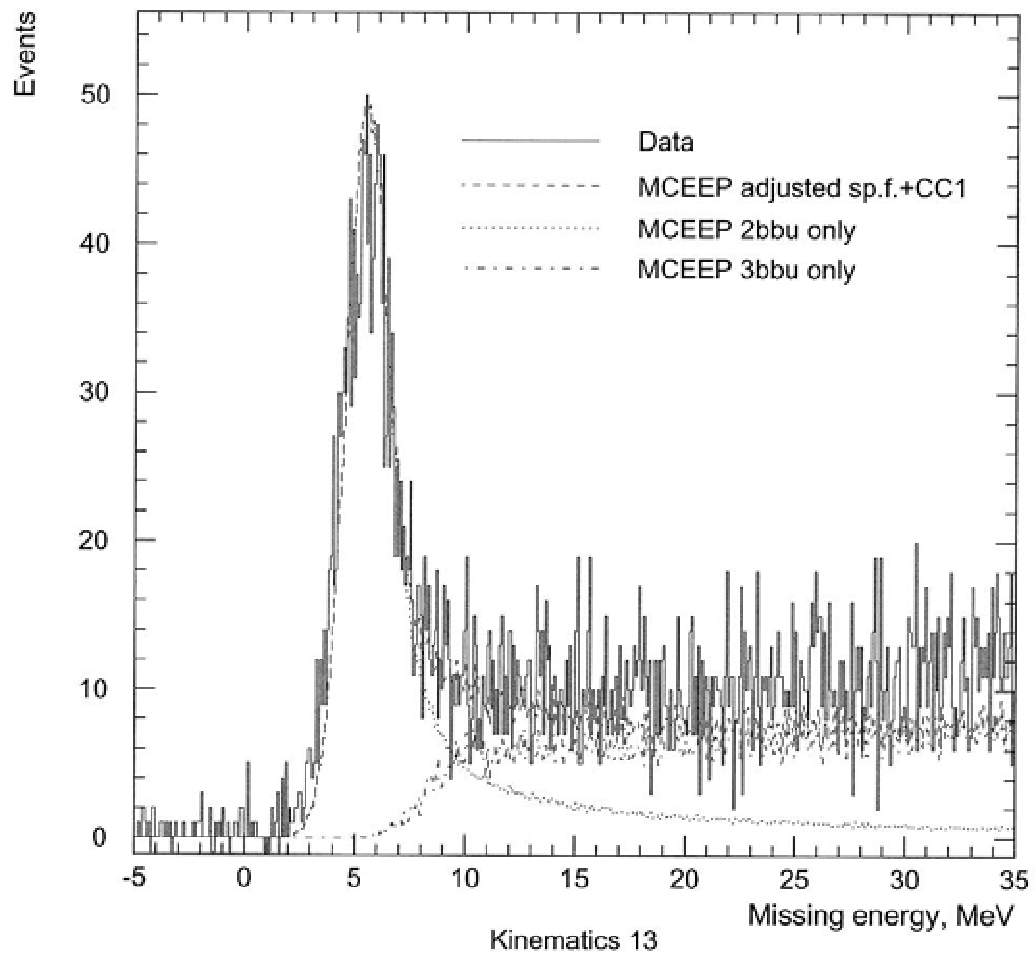


Figure 5-25: ^{3}He(e,e′p) missing energy distributions at kinematics 13, E_{beam} = 4.8 GeV, P_{miss} = 550 MeV/c, the detected proton is back of $\vec{q}$.