California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Real Analysis   Spring 2002

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Do **five** of the following problems. As follows.
The problems are divided into three groups:

- **A** Advanced calculus and classsical analysis
- **B1** Measure and integration
- **B2** Functional analysis

Selct problems as follows:

(1) Select at least two problems from part **A** (1-4).
(2) Select at least two problems from either part **B1** (5-8) or part **B2** (9-12).
(3) Select any fifth problem

Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation:
$\mathbb{R}$ denotes the set of real numbers.
$|z|$ denotes the absolute value of the number z.
$\{x_n\}_{n=1}^{\infty}$ denotes a sequence $x_1, x_2, x_3, \ldots$.
If A and B are sets, then $A \setminus B$ denotes the set difference $A \setminus B = \{x \in A : x \notin B\}$.
'Measure" refers to Lebesgue measure and $\mathbb{R}$, and "measurable functions" to Lebesgue measuurable functions.

PART A: Advanced Calculus and Classical Analysis

Spring 2001 # 1. Let $f : \mathbb{R} \to \mathbb{R}$ and for any integer $n > 0$ define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = f(x + \frac{1}{n})$. Prove that

a. If f is continuous then the sequence $\{f_n\}$ converges pointwise (simply) to f.
b. If f is uniformly continuous then the sequence $\{f_n\}$ converges uniformly to f.

Spring 2002 # 2. **a.** Suppose the completeness of the real line is characterized by the hypothesis that every nonempty subset of $\mathbb{R}$ which is bounded above has a least upper bound in $\mathbb{R}$

Use this property to prove that a monotonically increasing bounded sequence in $\mathbb{R}$ must converge to some real number.

b. Consider the sequence $\{a_n\}$ defined as follows: $a_1 = 2$ and $a_{n+1} = \dfrac{(2a_n + 5)}{6}$ for $n = 1, 2, 3, \ldots$.

i. Show that $1 \leq a_n \leq 2$ for all $n = 1, 2, 3, \ldots$.
ii. Prove that this sequence is convergent and find its limit.

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(Suggestion: It might be helpful to look at what you think the limit is in (ii) and then possibly reconsider the bounds found in (i).)

Spring 2002 # 3. Let $f : [0, \infty) \to \mathbb{R}$ be continuous with $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$. Prove that

a. f is bounded.
b. f is uniformly continuous.

Spring 2002 # 4. **a.** Give a statement of the intermediate value theorem and show how it follows from the fact that a subset of $\mathbb{R}$ is connected if and only if it is an interval (or a half line or the line).

b. Let $p(x)$ be a polynomial of odd degree with real coefficients. Show that the equation

$$p(x) = e^{-x^2}$$

has at least one real solution.

PART B1: Measure and Integration
"Measure" refers to Lebesgue measure and $\mathbb{R}$, and "measurable functions" to Lebesgue measuurable functions.

Spring 2002 # 5. **a.** Prove that every set with outer measure zero is measurable.

b. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$be measurable functions.

i. Show that $f + g$ and $f \cdot g$ are measurable
ii. Show that $\{x \in \mathbb{R} : (f(x))^2 < (g(x))^2\}$ is a measurable set.

Spring 2002 # 6. **a.** Let $\{E_n\}$ be a decreasing sequence of measurable sets with $\mu(E_n) < \infty$. Prove that

$$\mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n).$$

b. Let $[a, b]$ be a bounded interval in $\mathbb{R}$ and $\mathcal{C}$ be a collection of disjoint measurable subsets of $[a, b]$. Prove that the subcollection $\{A \in \mathcal{C} : \mu(A) > 0\}$ is countable.

Spring 2002 # 7. **a.** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions. Prove that $\limsup_n f_n$ and $\liminf_n f_n$ are both measurable.

b. Let f be an integrable function on the interval $I = [0, 1]$ and for each $n = 1, 2, 3, \cdots$ define f_n on I by

$$f_n(x) = \begin{cases} |f(x)|, & \text{if } |f(x)| \le n \\ n, & \text{if } |f(x)| > n \end{cases}$$

Prove that $\lim_{n\to\infty} \int_I f_n(x)dx = \int_I |f(x)|dx$.

Spring 2002 # 8. **a.** Suppose $f_1, f_2, f_3, \ldots$ is a sequence of integrable functions on an interval $[a, b]$ which converge pointwise to a limit function f. State carefully at least one thorem which gives additional conditions sufficient to justify the conclusion that

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

b. Give an example of a sequence of integrable functions on an interval $[a, b]$ (your choice of interval) which converge to a limit f such that $\lim_{n\to\infty} \int_a^b f_n(x)\, dx$ and $\int_a^b f(x)\, dx$ both exist but are not equal.

c. Evaluate $\lim_{n\to\infty} \int_0^2 \frac{x^n}{1 + x^n}\, dx$ giving reasons to justify your conclusion.

PART B2: Functional analysis

Spring 2002 # 9. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and $T : \mathcal{X} \to \mathcal{Y}$ be a linear operator. Prove that the following statements are equivalent:

(1) T is continuous at 0.

(2) T is continuous at each $x \in X$.

(3) T is bounded as a linear operator. (Also Define what this means)

Spring 2002 # 10. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator, and $\mathcal{Y}_0$ be a closed vector subspace of $\mathcal{Y}$. Show that $\mathcal{X}_0 = \{x \in \mathcal{X} : T(x) \in \mathcal{Y}_0\}$ is a closed vector subspace of X.

Spring 2002 #11. **a.** Carefully state any version of the Hahn-Banach Theorem.

b. Let $\mathcal{X}$ be a normed space and $x, y \in \mathcal{X}$. Prove that if $\phi(x) = \phi(y)$ for every continuous linear functional ϕ on $\mathcal{X}$ then $x = y$.

Spring 2002 # 12. Let $T : \mathcal{X} \to \mathcal{Y}$ be a linear operator. Assume that T maps $\mathcal{X}$ onto $\mathcal{Y}$ and that there is a constant $c > 0 \in \mathbb{R}$ such that $\| Tx \| \geq c \| x \|$ for every $x \in \mathcal{X}$. Prove that T has an inverse $T^{-1} : \mathcal{Y} \to \mathcal{X}$ which is linear and bounded as a linear operator.

End of Exam