California State University – Los Angeles
Department of Mathematics and Computer Science
Master's Degree Comprehensive Examination

Real Analysis   Spring 2001

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Do **five** of the following eight problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\{x_n\}_{n=1}^{\infty}$ denotes a sequence $x_1, x_2, x_3, \ldots$.
$(X, \mathcal{A})$ denotes a set X together with a σ-algebra of subsets of X.
$(X, \mathcal{A}, \mu)$ denotes a set X together with a σ-algebra of subsets of X and a non-negative measure μ defined on $\mathcal{A}$.
If A and B are sets, then $A \setminus B$ denotes the set difference $A \setminus B = \{x \in A : x \notin B\}$.

Spring 2001 # 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a real valued function on $\mathbb{R}$ such that $\lim_{x\to\infty} xf(x) = L$ with $L \in \mathbb{R}$. Show that $\lim_{x\to\infty} f(x) = 0$

Spring 2001 # 2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a real valued function on $\mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all x and y in $\mathbb{R}$. Assume that $\lim_{x\to 0} f(x) = L$ exists in $\mathbb{R}$.

a. Show that $L = 0$

(Suggestion: Consider $f(1) = f\left(n \cdot \frac{1}{n}\right)$)

b. Show f is continuous at every point of $\mathbb{R}$.

Spring 2001 # 3. **a.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $|f'(t)| \le M < \infty$ for all t in $\mathbb{R}$. Show that f is uniformly continuous on $\mathbb{R}$. (You may use basic theorems from calculus.)

b. Give an example of a uniformly continuous function on $[0, 1]$ which is differentiable on the open interval $(0, 1)$ but whose derivative is not bounded on $(0, 1)$. Justify your answer.

Spring 2001 # 4. Let (Ω, Σ) be a measurable space. (You may assume that Ω is the real line with Lebesgue measure if you wish.)

a. Define what it means for a function $f : \Omega \to \mathbb{R}$ to be measurable.

b. Us your definition to show that if $f : \Omega \to \mathbb{R}$ is a measurable function, then so is the function $f^2 : \Omega \to \mathbb{R}$ defined by $f^2(t) = (f(t))^2$ for all t in Ω.

c. Is the converse true? If f^2 is measurable must f be measurable?

Spring 2001 # 5. If f and g are (Lebesgue) measurable real-valued functions and $(f^2 + g^2)^{1/2}$ is integrable, prove that $f + g$ is (Lebesgue) integrable.

Spring 2001 # 6. For each $n = 1, 2, 3, \ldots$, define a function $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = \frac{1}{1 + x^{2n}}.$$

a. Explain how you know that f_n is integrable on $[0, \infty)$.

b. Show that $\lim\limits_{n\to\infty} \int_0^\infty \frac{1}{1 + x^{2n}}\, dx = 1.$

Spring 2001 # 7. Suppose h is an integrable function on a measurable domain E such that $h(t) \ge 0$ for almost every t in E. Show that if $0 < \beta < \infty$, then

$$\mu(\{t \in E \mid h(t) \ge \beta\}) \le \frac{1}{\beta} \int_E h(t)\, d\mu(t).$$

(You may assume that μ is Lebesgue measure on the real line if you wish.)

Spring 2001 # 8. Suppose A_1, A_2, A_3, … are measurable sets in a finite measure space (Ω, Σ, μ) such that $\sum_{k=1}^{\infty} \mu(A_k \setminus A_{k-1}) < \infty$ and $\lim_{n\to\infty} \mu(A_n) = 0$.

Prove that $\mu(\limsup A_k) = 0$

Note: (1) You may consider Ω to be a bounded interval in the real line with Lebesgue measure if you wish.

(2) $\limsup A_k = \{t \in \Omega : t \in A_k$ for infintely many indices $k\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

End of Exam