

California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination
Real and Functional Analysis Fall 2002
Beer, Chang*, Verona

(Retyped with some editing and corrections, 04/15/03, Hoffman)

Do **five** of the following problems. As follows.
The problems are divided into three groups:

- A** Advanced calculus and classical analysis
- B1** Measure and integration
- B2** Functional analysis

Select problems as follows:

- (1) Select at least two problems from part **A** (1-4).
- (2) Select at least two problems from either part **B1** (5-8) or part **B2** (9-12).
- (3) Select any fifth problem

Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation:

\mathbb{R} denotes the set of real numbers.

\mathbb{Z}^+ denotes the set of real numbers.

$|z|$ denotes the absolute value of the number z .

$\{x_n\}_{n=1}^{\infty}$ denotes a sequence x_1, x_2, x_3, \dots .

If A and B are sets, then $A \setminus B$ denotes the set difference $A \setminus B = \{x \in A : x \notin B\}$.

PART A: Advanced Calculus and Classical Analysis

Fall 2002 # 1. Let a_1, a_2, a_3, \dots and b_1, b_2, b_3, \dots be sequences of real numbers with $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$.

- a. Show that $\{a_n : n \in \mathbb{Z}^+\}$ is a bounded set. That is, there is a positive number r such that $|a_n| < r$ for all n in \mathbb{Z}^+ .
- b. Using boundedness, prove that $\lim_{n \rightarrow \infty} a_n b_n = ab$.

Fall 2002 # 2. Let A be a subset of \mathbb{R} and $f : A \rightarrow \mathbb{R}$.

- a. What does it mean for f to be uniformly continuous on A ?
- b. Using the definition given in **a**, show that if f is uniformly continuous on a bounded subset A of \mathbb{R} , then f is bounded on A .

Fall 2002 # 3. Let f_1, f_2, f_3, \dots be a sequence of real valued functions defined on \mathbb{R} , and $f : \mathbb{R} \rightarrow \mathbb{R}$.

- a. What does it mean for the sequence $\{f_n\}$ to converge pointwise to the function f ?
- b. What does it mean for the sequence $\{f_n\}$ to converge uniformly to the function f ?
- c. Prove that if f is a uniformly continuous function and $f_n(x) = f(x + (1/n))$ for each n in \mathbb{Z}^+ , then the sequence $\{f_n\}$ converges uniformly to f .

Fall 2002 # 4. a. Define what it means for a bounded real valued function f on a bounded interval $[a, b]$ to be Riemann integrable.

b. Using the definition given in **a**, prove that the following function is Riemann integrable on $[0, 1]$.

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1/2 \\ 1, & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

PART B1: Measure and Integration

Fall 2002 # 5. a. Let A_1, A_2, A_3, \dots be a sequence of sets. Define the sets $\limsup A_n$ and $\liminf A_n$.

b. For each positive integer n , let A_n be the closed interval $[-3/n, 2 - (1/n)]$ in \mathbb{R} . Find each of the following.

$$\bigcup_{k=n}^{\infty} A_k, \quad \bigcap_{k=n}^{\infty} A_k, \quad \limsup A_n, \quad \liminf A_n$$

Fall 2002 # 6. For each subset E of the real numbers \mathbb{R} , let $\mu^*(E) = 0$ if E is countable and $\mu^*(E) = 1$ if E is uncountable. Show that μ^* is an outer measure defined on the set of all subsets of \mathbb{R} , and determine the σ -algebra of the μ^* -measurable sets.

Fall 2002 # 9. Let (Ω, Σ, μ) be a measure space and A_1, A_2, A_3, \dots be a sequence of measurable sets.

- a. Show that there is a pairwise disjoint sequence B_1, B_2, B_3, \dots of measurable sets with $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$,
 - b. Prove that μ is countable subadditive.
-

Fall 2002 # 8. Let (Ω, Σ, μ) be a measure space. Let E_1, E_2, E_3, \dots be a sequence of pairwise disjoint measurable sets with $\bigcup_{n=1}^{\infty} E_n = \Omega$. Prove that if f is a nonnegative integrable function, then

$$\int_{\Omega} f d\mu = \sum_{n=1}^{\infty} \int_{E_n} f d\mu.$$

PART B2: Functional Analysis

Fall 2002 # 9. Let \mathcal{X} be normed space over \mathbb{R} . Let \mathcal{X}^* be the space of bounded linear functionals on \mathcal{X} . For f in \mathcal{X}^* , let $\|f\| = \sup\{f(v) : v \in \mathcal{X} \text{ and } \|v\| \leq 1\}$

- a. Prove that the supremum is finite.
 - b. Let \mathcal{X} be \mathbb{R}^2 with the Euclidean norm and $f((x, y)) = x + 2y$. Find $\|f\|$.
-

Fall 2002 # 10. Let \mathcal{X} be normed space over \mathbb{R} and $f : \mathcal{X} \rightarrow \mathbb{R}$ be a linear functional.

- a. Prove that $\mathcal{Y} = \{x \in \mathcal{X} : f(x) = 0\}$ is a subspace of \mathcal{X} .
 - b. Prove that f is continuous (bounded) if and only if \mathcal{Y} is a closed subspace of \mathcal{X} .
-

Fall 2002 # 11. a. Give the definitions of the sequence spaces ℓ_1 and ℓ_2 . That is, define the underlying sets, the vector structures, and the norms.

- b. Show that $\ell_1 \subseteq \ell_2$.
 - c. Show that the inclusion in part **b** is strict by finding a sequence which is in ℓ_2 but not in ℓ_1 .
-

Fall 2002 # 12. Let \mathcal{X} be normed space over \mathbb{R} and \mathcal{X}^* be the space of bounded linear functionals on \mathcal{X} with the norm defined in problem 9. For each x in \mathcal{X} , define $\hat{x} : \mathcal{X}^* \rightarrow \mathbb{R}$ by $\hat{x}(f) = f(x)$ for each $f \in \mathcal{X}^*$.

- a. Show that \hat{x} is linear and bounded (or continuous).
 - b. In view of part **a**, \hat{x} is a linear functional on \mathcal{X}^* and we can consider its norm, $\|\hat{x}\| = \sup\{\hat{x}(f) : f \in \mathcal{X}^* \text{ and } \|f\| \leq 1\}$. Show that $\|\hat{x}\| \leq \|x\|$.
-

End of Exam
