California State University – Los Angeles
Department of Mathematics and Computer Science
Master's Degree Comprehensive Examination

Real Analysis Fall 1999

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corrected 11/25/99 (Hoffman)

Do **five** of the following eight problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\{x_n\}_{n=1}^{\infty}$ denotes a sequence $x_1, x_2, x_3, \ldots$.
$(X, \mathcal{A})$ denotes a set X together with a σ-algebra of subsets of X.
$(X, \mathcal{A}, \mu)$ denotes a set X together with a σ-algebra of subsets of X and a non-negative measure μ defined on $\mathcal{A}$.
If A and B are sets, then $A \setminus B$ denotes the set difference $A \setminus B = \{x \in A : x \notin B\}$.

1. Define the limit superior and limit inferior of a sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers by

$$\overline{\lim}\, x_n = \inf_n \left(\sup_{k\geq n}\{x_k\} \right) \quad \text{and} \quad \underline{\lim}\, x_n = \sup_n \left(\inf_{k\geq n}\{x_k\} \right).$$

a. Suppose L is a finite real number. Show that $L = \overline{\lim}\, x_n$ if and only if the following two conditions hold.

(1) For each $\varepsilon > 0$ there is an index n such that $x_k < L + \varepsilon$ whenever $k \geq n$, and
(2) For each $\varepsilon > 0$ and each index n there is an index $k > n$ such that $x_k > L - \varepsilon$.

b. Show that $\overline{\lim}\, x_n = +\infty$ if and only if for each $\Delta > 0$ and each index n there is an index $k \geq n$ with $x_k > \Delta$.

c. Show that the sequence converges to a finite real number L if and only if $\overline{\lim}\, x_n = \underline{\lim}\, x_n = L$.

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2. Let $f(x) = x(2-x)$, and suppose that the sequence$\{x_n\}_{n=1}^{\infty}$ is defined recursively by

$$x_1 = \frac{1}{2} \quad \text{and} \quad x_{n+1} = f(x_n) \quad \text{for } n = 1, 2, 3, \ldots \quad .$$

a. Compute x_2 and x_3.
b. Show that $0 < x_n < 1$ for all $n = 1, 2, 3, \ldots$. (Hint: Use induction.)
c. Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ is monotone, that it converges, and find its limit.

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3. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be an $\mathcal{A}$-measurable function. Define the function $g : X \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } f(x) \text{ is rational} \\ 1 & \text{if } f(x) \text{ is irrational} \end{cases}.$$

Prove that the function g is $\mathcal{A}$-measurable.

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4. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Suppose A_1, A_2, A_3, $\ldots$ is a sequence of measurable sets with

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots \quad ,$$

and let $B = \bigcap_{n=1}^{\infty} A_n$.

Show that if $\sum_{n=1}^{\infty} \mu(A_n) < +\infty$, then $\mu(B) = 0$.

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5. Let $(X, \mathcal{A}, \mu)$ be a finite measure space Let f_1, f_2, f_3, $\ldots$ be a sequence of $\mathcal{A}$-measurable functions. Show that if $f_n \to f$ in measure and $f_n \to g$ in measure, then $f = g$ almost everywhere.

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6. Suppose c is a real constant and $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying the following conditions

(1) $f(t) > 0$ for all $t \geq 0$
(2) $f'(t)$ is continuous
(3) and $f'(t) \leq cf(t)$ for all $t \geq 0$.

Show that $f(x) \leq f(0)e^{cx}$ for $x \geq 0$.

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7. Suppose $f_1,\ f_2,\ f_3,\ \ldots$ are integrable functions on a measurable domain E with $m(E) < \infty$ and that $f_n \to f$ uniformly on E. Show that

a. Show that f is integrable on E.

b. Show that $\lim_{n\to\infty} \int_E f_n = \int_E f$.

(Suggestion: You might want to show that $|f(x)| \le |f_n(x)| + 1$ for large n.)

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8. Let m be Lebesgue measure on the real line.

(a) Let $A = \{x \in \mathbb{R} \mid \cos 3x = 0\}$. Show that A is measurable and find $m(A)$.

(b) Let B be the set of all real numbers x for which there is an integer n such that $\cos nx = 0$. Find $m(B)$.

(c) Show that there are uncountably many real numbers t with $0 < t < \pi/2$ such that $\cos nt$ is not equal to 0 for any integer n.

(That is: $\exists$ uncountably many t with $0 < t < \pi/2$ such that $\forall n \in \mathbb{Z}, \cos nt \neq 0$.)

End of Exam