

California State University – Los Angeles
Department of Mathematics and Computer Science
Master's Degree Comprehensive Examination
Real Analysis Fall 1997
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Retyped 4/1/00 (Hoffman)

Do **five** of the following eight problems. Each problem is worth 20 points.

1. Let X be a nonempty set, and let \mathcal{A} be the family of subsets of X that are either finite or have finite complement.

- a. Show that \mathcal{A} is an algebra of sets
 - b. Show that if X is infinite, then \mathcal{A} is not a sigma algebra of sets.
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2. Let $f_i : \mathbb{R} \rightarrow \mathbb{R}$ be Lebesgue measurable functions for $i = 1, 2, 3$. Prove that the following set is measurable: $\{x \in \mathbb{R} : f_1(x) < f_2(x) < f_3(x)\}$.

3. Let $\langle a_n \rangle_{n=1}^{\infty}$ be a sequence of real numbers. Prove or give a counterexample to each of these statements.

- a. If $\sum_{n=1}^{\infty} |a_n| < \infty$, then $\sum_{n=1}^{\infty} a_n^2 < \infty$.
 - b. If $\sum_{n=1}^{\infty} a_n^2 < \infty$, then $\sum_{n=1}^{\infty} |a_n| < \infty$.
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4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function, and define $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{if } f(x) \text{ is rational} \\ 0 & \text{if } f(x) \text{ is irrational} \end{cases} .$$

Prove that g is a measurable function.

5. a. State Fatou's Lemma and the Lebesgue Dominated Convergence Theorem.
 - b. Use Fatou's Lemma to prove the Dominated Convergence Theorem.
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6. Let f and g be Lebesgue measurable functions on a measurable set A . Using standard properties of integrable functions, prove that their maximum h defined by $h(x) = \max\{f(x), g(x)\}$ is also integrable on A .

7. For each of the following, give an example (*without justification*).

a. A bounded function $f : [0, 1] \rightarrow \mathbb{R}$ which is not Riemann integrable.

b. Real valued continuous functions f, f_1, f_2, f_3, \dots on the interval $[0, 1]$ such that the sequence $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwise to f on $[0, 1]$ but $\int_0^1 f(x) dx \neq \lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$

c. A continuous function that is not uniformly continuous. (Specify the domain.)

d. A G_δ subset of \mathbb{R} that is neither open nor closed.

e. A function $f : [0, 1] \rightarrow \mathbb{R}$ which has infinitely many points of discontinuity yet is still Riemann integrable.

8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function and let a be a fixed point in \mathbb{R} . For each $\delta > 0$ let $\delta_f(a)$ be defined as

$$\delta_f(a) = \sup\{|f(x) - f(w)| : |x - a| < \delta \text{ and } |w - a| < \delta\}.$$

and the *oscillation* of f at a by

$$\text{osc}_f(a) = \lim_{\delta \rightarrow 0^+} \delta_f(a).$$

a. For the function f defined by

$$f(x) = \begin{cases} 4 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x & \text{if } x > 1 \end{cases}$$

find $\delta_f(1)$ for each $\delta > 0$ and find the oscillation of f at 1.

b. Prove that if f is continuous at a , then $\text{osc}_f(a) = 0$. (The converse is also true, but you do not have to prove that.)

End of Exam