Part B1: Measure and Integration

In the sequel a *measure space* (X, Σ, μ) is a set X equipped with a sigma-algebra of subsets Σ and a countably additive measure μ defined on Σ.

5. Let (X, Σ, μ) be a measure space.

   (a) What does it mean for a sequence of measurable functions (f_n) on X to converge in measure to a function f?

   (b) Let (f_n) and (g_n) be sequences of measurable functions that converge in measure to f and g, respectively. Using the definition given in (a), prove that the sequence $(f_n - g_n)$ converges in measure to $f - g$.

   (c) Give an example of a pointwise convergent sequence of Lebesgue measurable functions that does not converge in measure.

6. Let (X, Σ, μ) be a measure space. Let Σ_0 be this subfamily of Σ:

$$\Sigma_0 = \{A \in \Sigma : \text{ either } \mu(A) = 0 \text{ or } \mu(A^c) = 0\}.$$

   Is Σ_0 an algebra of subsets of X? Is Σ_0 a σ-algebra of subsets of X? Justify your answers.

7. Let (X, Σ, μ) be a finite measure space.

   (a) Suppose (A_n) is a decreasing sequence in Σ. Prove that $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

   (b) Let $f: X \to \mathbb{R}$ be measurable, and let $A_n = \{x : f(x) > n\}$. Prove $\lim_{n\to\infty} \mu(A_n) = 0$.

8. Let (X, Σ, μ) be a measure space, and let f be a nonnegative integrable function. For each positive integer n let $f_n(x) = f(x)$ when $f(x) \leq n$ and $f_n(x) = 1/n$ when $f(x) > n$.

   (a) Explain carefully why each function f_n is both measurable and integrable.

   (b) Prove $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$