Real and Functional Analysis Comprehensive Exam
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Do five of the following problems. Each is worth 20 points. The problems are divided into 3 groups:

A Advanced calculus and classical analysis
B1 Measure and integration
B2 Functional analysis

Select at least two problems from part **A** (1-4). Select at least two from either part **B1** (5-8) or from part **B2** (9-12). Finally, select any fifth problem. In what follows $\mathbb{N}$ denotes the positive integers, $\mathbb{R}$ denotes the real numbers, and $\mathbb{C}$ denotes the complex numbers.

Part A: Advanced calculus and classical analysis

1. Let A be a closed subset of $\mathbb{R}$ and let (a_n) be a Cauchy sequence in A. Prove that (a_n) has a subsequence convergent to a point of A. You may not assume that the usual metric is complete, as that is what in part you are asked to prove! But you may assume the Bolzano-Weierstrass Theorem.

2. Is the following function continuous on $\mathbb{R}$? Justify your answer carefully.

$$f(x) = \sum_{n=1}^{\infty} \frac{(n+1)\sin(nx)}{n!} x^2$$

3. (a) State the Mean Value Theorem.
   (b) Suppose S is an open interval, an open ray, or $\mathbb{R}$ itself and $f: S \to \mathbb{R}$. Show that if f is differentiable on S and $\forall\ x \in S\ f'(x) = 0$, then f is constant on S.
   (c) Prove that $\tan x > x$ for each $x \in (0,\pi/2)$.

4. Check if the following functions are Riemann Integrable on the interval $[-10,10]$. You may use any definition of Riemann integrability you like as well as any theorems that could be applied to justify your assertions.

(a) $$f(x) = \begin{cases} x & \text{if } -10 < x < -\pi \\ \sin x & \text{if } -\pi < x < \pi \\ \dfrac{e^x}{x} & \text{if } \pi \le x \le 10 \end{cases}$$

(b) $$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$