

# California State University – Los Angeles

## Mathematics

### Masters Degree Comprehensive Examination

**Real and Functional Analysis      Fall 2008**  
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Do **five** of the following problems as follows. The problems are divided into three groups.

**A** : Advanced Calculus and Classical Analysis

**B1** : Measure and Integration

**B2** : Functional Analysis

Select Problems as follows:

- Select at least two problems from part **A**.
- Select at least two problems from either part **B1** or part **B2** (same section).
- Select any fifth problem

All problems count equally. If you attempt more than the minimum, we will take the best five meeting the selection criteria above.

Please

- (1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
- (2) Write on one side of the paper only
- (3) Begin each problem on a new page
- (4) Assemble the problems you hand in in numerical order

**Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.**

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Notation:  $\mathbb{C}$  denotes the set of complex numbers.

$\mathbb{R}$  denotes the set of real numbers.

$\mathbb{N}$  denotes the set of positive integers (natural numbers).

$|z|$  denotes the absolute value of the complex number  $z$

$\{x_n\}$  or  $\{x_n\}_{n=1}^{\infty}$  denotes a sequence  $x_1, x_2, x_3, \dots$

If  $A$  and  $B$  are sets, then  $A \setminus B$  is the set difference,  $A \setminus B = \{x \in A : x \notin B\}$ .

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**Part A: Advanced Calculus and Classical Analysis**


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**Fall 2008 # 1.** Let  $\{x_n\}_{n=1}^{\infty}$  be a bounded sequence of real numbers. For each  $n$ , let

$$s_n = \sup\{x_k : k \geq n\} \quad \text{and} \quad S = \inf\{s_n : n \geq 1\}.$$

Show that there is a subsequence of  $\{x_n\}_{n=1}^{\infty}$  which converges to  $S$ .

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**Fall 2008 # 2.** Suppose  $f_1, f_2, f_3, \dots$  are real valued functions on a subset  $A$  of  $\mathbb{R}$  and  $f : A \rightarrow \mathbb{R}$ .

- a. Define what it means for the sequence  $\{f_n\}$  to converge to  $f$  uniformly on  $A$ .
  - b. Define what it means for a function  $g : A \rightarrow \mathbb{R}$  to be uniformly continuous on  $A$ .
  - c. Show that if each  $f_n$  is uniformly continuous on  $A$  and  $f_n \rightarrow f$  uniformly on  $A$ , then  $f$  must also be uniformly continuous on  $A$ .
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**Fall 2008 # 3. a.** Give a statement of the mean value theorem for derivatives.

**b.** Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable function and that  $f'(x) \rightarrow b \in \mathbb{R}$  as  $x \rightarrow +\infty$ , and let  $h$  be a fixed positive real number. Show that

$$\lim_{x \rightarrow \infty} \left( \frac{f(x+h) - f(x)}{h} \right) = b$$


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**Fall 2008 # 4. a.** State a definition (or an equivalent condition) for what it means for a bounded real valued function on a closed bounded interval  $[a, b]$  to be Riemann integrable on that interval.

**b.** Use the condition given in **a** to show that the function defined on  $[0, 1]$  by  $f(x) = 0$  if  $x$  is irrational and  $f(x) = x$  if  $x$  is rational is not Riemann integrable.

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**Part B1: Measure and Integration**


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**Fall 2008 # 5.** Let  $X$  be a non-empty set. Let  $\mathcal{A}$  be the family of subsets of  $X$  that are either finite or have finite complement.

- (a) Show that  $\mathcal{A}$  is an algebra of sets.
- (b) Show that if  $X$  is infinite, then  $\mathcal{A}$  is not a  $\sigma$ -algebra.

Let  $\mathcal{B}$  be the set of all subsets of  $X$  that are either countable or have countable complement. (A finite set is countable.)

- (c) Is  $\mathcal{B}$  a  $\sigma$ -algebra? Why or why not ?
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**Fall 2008 # 6.** Evaluate  $\lim_{n \rightarrow \infty} \int_1^{\infty} \frac{(x+n) \cos(1/(nx))}{(3x+2n)x^2} dx$  Giving justification for your result.

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**Fall 2008 # 7.** Let  $\Sigma$  be a  $\sigma$ -algebra of subsets of a domain  $\Omega$ .

- a. Define what it means for a function  $f : \Omega \rightarrow \mathbb{R}$  to be  $\Sigma$ -measurable
- b. Suppose  $f : \Omega \rightarrow \mathbb{R}$  is  $\Sigma$ -measurable. Let  $g : \Omega \rightarrow \mathbb{R}$  be given by

$$g(x) = \begin{cases} 1/f(x) & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

Use the definition you gave in part **a** to show that  $g$  is  $\Sigma$ -measurable.

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**Fall 2008 # 8.** Let  $(\Omega, \Sigma, \mu)$  be a measure space.

- a. Define what it means for a sequence  $\{f_n\}$  of  $\Sigma$ -measurable functions on  $\Omega$  to converge in measure to a function  $f$ .
  - b. Suppose  $\mu(\Omega) < \infty$ . Let  $\{f_n\}$  and  $\{g_n\}$  be sequences of functions converging in measure to functions  $f$  and  $g$  respectively. Show using the definition from part **a** that if  $\alpha$  and  $\beta$  are real numbers, then the sequence  $\{\alpha f_n + \beta g_n\}$  converges in measure to  $\alpha f + \beta g$ .
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**Part B2: Functional Analysis**


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**Fall 2008 # 9.** Suppose  $v_1, v_2, v_3, \dots$ , is a sequence of vectors in a Banach space  $\mathcal{X}$  with norm  $\|\cdot\|$ . Show that if the series  $\sum_{k=1}^{\infty} \|v_k\|$  converges to a finite sum in  $\mathbb{R}$ , then the series  $\sum_{k=1}^{\infty} v_k$  converges in  $\mathcal{X}$ . (That is, show that absolute convergence implies convergence in  $\mathcal{X}$ .)

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**Fall 2008 # 10.** Let  $\phi$  be a nonzero linear functional on a vector space  $\mathcal{X}$  over  $\mathbb{R}$ .

Let  $N(\phi) = \text{kernel}(\phi)$  be the null space or kernel of  $\phi$ .

Let  $v_o$  be in  $\mathcal{X} \setminus N(\phi)$ .

- Show that each  $w$  in  $\mathcal{X}$  has a unique representation as  $v + tv_o$  with  $v$  in  $N(\phi)$  and  $t$  in  $\mathbb{R}$ .
  - Show that if  $\phi$  and  $\psi$  are nonzero linear functionals on  $\mathcal{X}$  with  $N(\phi) = N(\psi)$ , then there is a constant  $c$  with  $\phi(v) = c\psi(v)$  for all  $v$  in  $\mathcal{X}$ .
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**Fall 2008 # 11.** For each of the following, decide whether it defines a norm on  $\mathbb{R}^2$ . If it does, prove it or explain how you know; if it does not, explain why not.

- For  $(a, b) \in \mathbb{R}^2$ , put  $\|(a, b)\| = 3|a| + 2|b|$ .
  - For  $(a, b) \in \mathbb{R}^2$ , put  $\|(a, b)\| = 3|a| - 2|b|$ .
  - For  $(a, b) \in \mathbb{R}^2$ , put  $\|(a, b)\| = |a - b|$ .
  - For  $(a, b) \in \mathbb{R}^2$ , put  $\|(a, b)\| = \int_0^1 |a + bt| dt$ .
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Suggestion: You may, but do not have to, use the known fact that  $\int_0^1 |f(t)| dt$  gives a norm on  $\mathcal{C}([0, 1])$ .

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**Fall 2008 # 12.** Recall that two norms on a vector space  $\mathcal{V}$  are said to be equivalent if there are constants  $A > 0$  and  $B > 0$  such that

$$A \|v\|_a \leq \|v\|_b \leq B \|v\|_a \quad \text{for all } v \in \mathcal{V}.$$

**a.** Prove that if  $\|\cdot\|_a$  and  $\|\cdot\|_b$  are equivalent norms on  $\mathcal{V}$ , then a sequence  $\{f_n\}_{n=1}^{\infty}$  in  $\mathcal{V}$  converges with respect to  $\|\cdot\|_a$  if and only if it converges with respect to  $\|\cdot\|_b$ .

**b.** Prove that if  $\|\cdot\|_a$  and  $\|\cdot\|_b$  are equivalent norms on  $\mathcal{V}$ , then  $\mathcal{V}$  is complete with respect to  $\|\cdot\|_a$  if and only if it is complete with respect to  $\|\cdot\|_b$ .

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## End of Exam