Department of Mathematics
California State University, Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
SPRING 2004

Instructions: Do any **2** problems from Part **I** AND any **2** problems from Part **II**

PART I (Do two problems)

I-1 **a**. Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$.

Find the LU decomposition of the matrix B; i.e., find a unit lower triangular matrix L and an upper triangular matrix U such that $B = LU$.

b. Given the linear system $A\mathbf{x} = \mathbf{b}$, where A has been factored as $A = LU$, and

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}:$$

Solve $A\mathbf{x} = \mathbf{b}$ by forward/backward substitution (*without multiplying* L and U).

c. Determine the LDU decomposition of the matrix A of part ***b***; i.e., determine unit lower and upper triangular matrices L and U and diagonal matrix D such that $A = LDU$.

I-2 **a.** Let $B = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$.

By computing the eigenvalues of the iteration matrix, determine whether or not Jacobi iteration converges in solving $B\mathbf{x} = \mathbf{c}$, where $\mathbf{c}$ is an arbitrary 3-vector.

b. Given the linear system $A\mathbf{x} = \mathbf{b}$, where A is an $n \times n$ matrix and $\mathbf{b}$ is an arbitrary n-vector, write $A = N - P$, where N is nonsingular, and consider the iterative method:

$$\mathbf{x}^{(k+1)} = (N^{-1}P)\mathbf{x}^{(k)} + N^{-1}\mathbf{b} \quad (k = 0, 1, 2, 3, \ldots)$$

(i) Show that the error in the approximation satisfies:

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| \le \|M\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|, \quad \text{where } M = N^{-1}P \text{ and } \mathbf{x}^{(0)} \text{ is the initial vector}$$

(ii) Let ρ be the spectral radius of the matrix $M = N^{-1}P$, and assume that $\rho < 1$. Use the result of part **(i)** to show that if m is a positive integer and $k \ge m/(-\log_{10} \rho)$, then $\|\mathbf{x}^{(k)} - \mathbf{x}\| \le C(10^{-m})$, where C is a constant that is independent of k and m.

I-3 **a.** Let A be an $n \times n$ matrix with eigenvalues λ_k that satisfy $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \ldots \ge \lambda_n$. Let $\mathbf{X}_0$ be an initial vector and define $\mathbf{X}_{i+1} = A\mathbf{X}_i$ for $i = 0, 1, 2, \ldots$ What conditions on A and $\mathbf{X}_0$ guarantee that the sequence $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \ldots$ (with proper scaling) converges to the eigenvector corresponding to λ_1?

b. Let B be an $n \times n$ matrix.

(i) Briefly describe how to compute the matrices $B_1, B_2, B_3, \ldots$ in the QR algorithm for finding the eigenvalues of B.

(ii) Show that the matrices B_j and B_{j+1} of part **(i)** are similar matrices for each $j \ge 1$.

(iii) Prove that if C and D are similar $n \times n$ matrices, then C and D have the same eigenvalues.

PART II (Do two problems)

II-1 Consider the parabolic PDE $u_t = au_{xx} + u$, where $a > 0$.

a. Approximate the given PDE by an explicit finite-difference scheme (replacing u_t and u_{xx} by the usual forward and central differences, respectively).

b. Determine the local truncation error for the scheme of part **a**.

c. Assuming zero boundary conditions, obtain the $(n - 1) \times (n - 1)$ matrix A so that the scheme of part **a** is written in the form $\mathbf{u}^{(k+1)} = A\mathbf{u}^{(k)}$, where $\mathbf{u}^{(k)}$ represents the values of the approximate solution for $t = k\Delta t$.

d. Use the result of part **c** to show that the finite-difference scheme of part **a** is stable for $r \le 1/(2a)$, where $r = \Delta t/(\Delta x)^2$.

II-2 Consider the initial-boundary value problem:

$$\begin{cases} u_{tt} = (1/4)u_{xx} + 3u_x + u & 0 < x < 3, \quad t > 0 \\ u(x,0) = x^2, \quad u_t(x,0) = 3x & 0 \le x \le 3 \\ u(0,t) = 0, \quad u(3,t) = 0, & t > 0 \end{cases}$$

Suppose that we place a mesh (grid) on the region with spacing $h = \Delta x$, $k = \Delta t$, and let $u_{i,j} = u(i\Delta x, j\Delta t)$.

a. Derive an explicit scheme to obtain $u_{i,j+1}$ from $u_{i,j}$ and $u_{i,j-1}$.

b. Letting $\Delta x = \Delta t = 1$, use the scheme of part **a**, together with the initial and boundary conditions, to find u_{11}, the approximation to $u(1, 1)$.

c. Find the point of intersection, R, of the characteristic curves for the given PDE through the points P(1, 0) and Q(2, 0).

d. What restriction on the ratio k/h is given by the slopes of the characteristic curves?

II-3 Consider the following boundary-value problem on the "unit square":

$$u_{xx} + u_{yy} = 0 \qquad (0 < x < 1,\ 0 < y < 1)$$
$$u(0, y) = 0,\ u(1, y) = 1 - 3y^2 \qquad (0 \le y \le 1)$$
$$u(x, 0) = x^3,\ u(x, 1) = x^3 - 3x \qquad (0 \le x \le 1)$$

a. Verify that $u(x, y) = x^3 - 3xy^2$ is a solution of the given boundary-value problem.

b. Prove that $u(x, y) = x^3 - 3xy^2$ is the *only* solution to the given problem. (You may assume the minimum and maximum principles for the given PDE.)

c. Suppose that we take $h\ (= \Delta x = \Delta y) = 1/10$, and at each of the resulting interior grid points (x, y), we replace $u_{xx} + u_{yy} = 0$ by:

$$\frac{u(x-h,y) - 2u(x,y) + u(x+h,y)}{h^2} + \frac{u(x,y-h) - 2u(x,y) + u(x,y+h)}{h^2} = 0$$

Show that this finite-difference equation is consistent with the PDE $u_{xx} + u_{yy} = 0$.

d. Explain why the system of linear algebraic equations, $A\mathbf{u}_h = \mathbf{b}$, that results from applying the finite-difference scheme in part **c** has a unique solution. (You may assume the minimum and maximum principles for the discrete equation.)

e. Suppose that $A\mathbf{u}_h = \mathbf{b}$ (see part **d**) is solved exactly. Would the resulting finite-difference solution be *equal* to the continuous (exact) solution at every grid point? Explain your answer.