Department of Mathematics
California State University, Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
SPRING 2003

Instructions: Do any **2** problems from Part **I** AND any **2** problems from Part **II**

PART I (Do two problems)

I-1 Let A be the $n \times n$ matrix for which every entry is equal to 1.

a. Show that A is not invertible.

b. Solve the linear system $A\mathbf{x} = \mathbf{0}$ by Gaussian elimination.

c. Suppose we apply Gauss-Seidel iteration to the system $A\mathbf{x} = \mathbf{0}$ using the initial vector $\mathbf{x}^{(0)} = (1, 1, ..., 1)$. Find $\mathbf{x}^{(1)}$.

d. Show that Jacobi iteration diverges for $A\mathbf{x} = \mathbf{b}$, for arbitrary **b** and all n. [Hint: Show that the Jacobi iteration matrix has the eigenvalue -1 for all n.]

e. Suppose that the Power Method is used to find the eigenvector of A corresponding to the largest (in magnitude) eigenvalue. If the initial vector is $\mathbf{x}^{(0)} = (1, 0, ..., 0)$, find $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

I-2 **a.** Let A be the symmetric matrix $\begin{bmatrix} d & -1 & 0 \\ -1 & d & -1 \\ 0 & -1 & d \end{bmatrix}$, where $d > 2$.

Show that A is positive definite.

b. For the matrix A given in part **a**, find the decomposition A = LU, where L is unit lower-triangular and U is upper-triangular.

c. Prove the following: If B is an $n \times n$ symmetric positive definite matrix, then
(i) A is nonsingular, and
(ii) the diagonal elements of A satisfy $a_{kk} > 0$ for $k = 1, 2, ..., n$.

I-3 **a.** Let A be an $n \times n$ symmetric matrix with eigenvalues λ_i, and suppose that $\max |\lambda_i| < 1$ for $i = 1, 2, ..., n$. Prove that for every vector **v** in $\mathbf{R}^n$, $\lim A^k\mathbf{v} = \mathbf{0}$, as $k \to \infty$.

b. Let B be an $m \times n$ matrix with the property that if $B\mathbf{x} = \mathbf{0}$ for a vector **x**, then $\mathbf{x} = \mathbf{0}$. Does this imply that the columns of B are linearly independent? Explain your answer.

c. Suppose that the null space of an $m \times n$ matrix C contains a nonzero vector. Show that if $C\mathbf{x} = \mathbf{b}$ has a solution for some **b**, then this solution is not unique.

PART II (Do two problems)

II-1 **a.** What does it mean to say that a second-order, linear partial differential equation is elliptic, hyperbolic, or parabolic? Give a specific example of each type of equation.

b. Consider the hyperbolic partial differential equation

$$aU_{xx} + bU_{xy} + cU_{yy} + e = 0$$

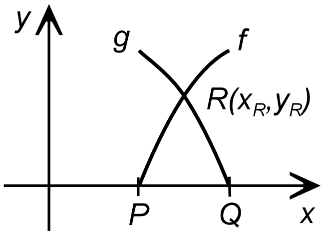
where a, b, c, and e are continuous functions of x, y, U, U_x, and U_y, and initial data is given on some non-characteristic curve Γ. Show that the characteristic curves on Γ satisfy the differential equation

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$$

c. Suppose that you use the method of characteristics to obtain an approximate solution to:

$$\begin{aligned} &U_{xx} - U_{yy} = 0 && 0 \le x \le 5,\ y > 0 \\ &U(x, 0) = x^2 && 0 \le x \le 5 \\ &U_y(x, 0) = x && 0 \le x \le 5 \\ &U(0, y) = U(5, y) = 3 && y \ge 0 \end{aligned}$$

at the grid point $R(x_R, y_R)$, which is the intersection of the characteristic curves through P(1, 0) and Q(2, 0). (See the figure; f and g are the characteristic curves through P and Q, respectively.)



(i) Find exact values for x_R and y_R.

(ii) Obtain approximate values for $U_R = U(x_R, y_R)$, $p_R = U_x(x_R, y_R)$, and $q_R = U_y(x_R, y_R)$. (Just get first approximations for these quantities; do not iterate.)

II-2 Suppose that u(x, y) satisfies the following differential equation and boundary conditions:

$$9u_{xx} + u_{yy} = 0, \quad 0 < x < 1,\ 0 < y < 1$$
$$u(0, y) = -9y^2,\ u(1, y) = 1 - 9y^2, \quad 0 \le y \le 1$$
$$u(x, 0) = x^2,\ u(x, 1) = x^2 - 9, \quad 0 \le x \le 1$$

a. Using the usual central difference formula to approximate the partial derivatives, find the finite-difference scheme for the given differential equation. *Simplify your answer.*

b. Derive the local truncation error for the finite-difference scheme from part **a**.

c. Use the finite-difference scheme from part **a** with $\Delta x = \Delta y = 1/3$ to approximate u(x, y) at the resulting four interior grid points. Simplify your answer and write it in the form of a linear system, A**u** = **b**. Do *not* solve this system.

II-3 Given the initial-boundary value problem:

$$\begin{cases} u_t = u_{xx} & 0 \le x \le 1, \quad t > 0 \\ u(x,0) = f(x), & 0 \le x \le 1, \quad (f(x) \text{ given}) \\ u(0,t) = 0, \quad u(1,t) = 0, & t > 0 \end{cases}$$

Suppose we approximate the PDE $u_t = u_{xx}$ by the finite-difference scheme

$$-u_{i-1,j+1} + 3u_{i,j+1} - u_{i+1,j+1} = u_{i,j}$$

where $u_{i,j} = u(ih, jk)$.

a. Is the given scheme explicit or implicit? Briefly explain why.

b. If the given scheme is written as $B\mathbf{u}_{j+1} = C\mathbf{u}_j$, where $\mathbf{u}_j = (u_{1,j}, u_{2,j}, \ldots, u_{N-1,j})$, determine the matrices B and C.

c. Explain (in one sentence each) what it means when we say that:

- The given scheme is *stable.*
- The given scheme is *consistent* with the given initial-boundary value problem.
- The given scheme *converges.*

d. Prove that the given scheme is stable.