Department of Mathematics and Computer Science
California State University, Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
SPRING 2001

Instructions: Do any **2** problems from Part **I** AND any **2** problems from Part **II**

PART I (Do two problems)

I-1 Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Note that A is symmetric and nonsingular (det(A) = -1).

a. Is A: (answer *yes* or *no* for each; no explanation required)

i. Diagonally dominant? **ii.** Diagonalizable?
iii. Positive definite? **iv.** Orthogonal?

b.. Give the value of $\|A\|_\infty$.

c. Apply Gershgorin's Theorem to A to locate its eigenvalues; that is, to determine the interval in which all eigenvalues lie.

d. Find the Jacobi iteration matrix and use it to determine whether or not Jacobi iteration converges for the given matrix A.

I-2 **a.** Let U be a nonsingular upper-triangular $n \times n$ matrix and let $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n$ be arbitrary n-vectors. Determine the number of multiplications/divisions needed to solve the n linear systems $U\mathbf{x}_k = \mathbf{b}_k$ ($k = 1, 2, ..., n$) using backward substitution.

b. Let U be a nonsingular upper-triangular $n \times n$ matrix. We can find the inverse of U by solving the matrix equation UX = I (where I is the identity matrix of order n) for X. Notice that the equation UX = I can be viewed as n linear systems $U\mathbf{x}_k = \mathbf{b}_k$, where the $\mathbf{b}_k$ are the columns of I. Determine the number of multiplications/divisions needed to solve these systems by backward substitution.

c. Use the method described in part **b** to find the inverse of the matrix

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

I-3 The matrix $A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1, \lambda_2 = 3$ with corresponding eigenvectors $\mathbf{x}_1 = (3, -1)$ and $\mathbf{x}_2 = (1, 1)$.

a. Find the QR factorization of A, where Q is orthogonal and R is upper-triangular.

b. Find the matrix resulting from performing one iteration of the QR method (for approximating eigenvalues) on A.

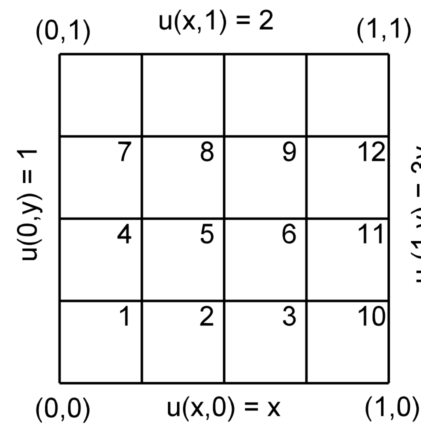
c. Perform *two* iterations of the Power Method on A, with $\mathbf{u}_0 = (4, 1)$, to find a vector, $\mathbf{u}_2$, approximating $\mathbf{x}_2$.

PART II (Do two problems)

II-1 Suppose that the function $u(x, y)$ satisfies

$$x u_{xx} + y u_{yy} = 3 \qquad (*)$$

inside the square region shown at the right (with $h = \Delta x = \Delta y = 1/4$), and that it satisfies the indicated boundary conditions.



a. Show that the partial differential equation (*) is elliptic in this region.

b. Denoting the finite difference solution at grid point k by u_k, ($k = 1, 2, \ldots, 12$), write finite difference equations consistent with (*) at grid points 4 and 5. (You do not have to show consistency.)

c. For a function $v(x, y)$, find constants c_0, c_1, and c_2 such that

$$\left| \frac{\partial v}{\partial x}(x_0, y_0) - [c_0 v(x_0, y_0) + c_1 v(x_0 - h, y_0) + c_2 v(x_0 - 2h, y_0)] \right| < K h^2$$

where K is a constant independent of h. (You may assume that $v(x, y)$ has as many partial derivatives as you need.)

d. Write an $O(h^2)$ finite difference equation to replace the boundary condition $u_x(1, 1/2) = 3y$ at grid point 11.

II-2 Consider the hyperbolic partial differential equation

$$a u_{xx} + b u_{xy} + c u_{yy} + e = 0, \qquad -\infty < x < \infty,\ y > 2$$

where a, b, c, d, and e may be functions of x, y, u, u_x, and u_y. Suppose that

$$u(x, 2) = f(x) \quad \text{and} \quad u_y(x, 2) = g(x) \qquad -\infty < x < \infty,$$

where f(x) and g(x) are given functions.

a. Show that at each point on the line y = 2, the characteristic curves satisfy:

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$$

b. Consider the initial-value problem

$$\begin{cases} u_{xx} = u^2 u_{yy} & -\infty < x < \infty,\ y > 2 \\ u(x, 2) = 2x & -\infty < x < \infty \\ u_y(x, 2) = x & -\infty < x < \infty \end{cases}$$

Let R be the point of intersection of the characteristic curves through the points P(1, 2) and Q(2, 2). Use the method of characteristics to find the first approximation to the coordinates of R.

c. Suppose that the initial data u(x, 2) = f(x) is discontinuous at the point (1, 2). What is the effect of the discontinuity on the solution u(x, y) for y > 2 ?

II-3 Consider the following difference approximation to the PDE

$$u_t = a u_{xx} + b u_x \qquad 0 < x < 1,\ t > 0 \qquad \text{(a and b are constants with } a > 0)$$
$$u(x, 0) = f(x) \qquad 0 < x < 1 \qquad \text{(f(x) is given)}$$
$$u(0, t) = 0,\ u(1, t) = 0 \qquad t > 0$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = a\left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right] + b\left[\frac{u_{i+1,j} - u_{i,j}}{h}\right],$$

where $u_{i,j} = u(ih, jk)$, $h > 0$, $k > 0$.

a. Is this an explicit or implicit scheme? Explain your answer.

b. Show that this scheme is *consistent* with the given PDE.

c. Let b = 0 in both the PDE and the finite difference scheme. For fixed k and h, determine the values of $r = k/h^2$ for which the scheme is *stable*.