Department of Mathematics
California State University, Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
FALL 2007

Instructions: Do any **2** problems from Part **I** AND any **2** problems from Part **II**

PART I (Do two problems)

I-1 **a**. Let $A = \begin{bmatrix} 1 & k & k \\ k & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$, where $k > 0$.

(i) Show that the spectral radius of the *Jacobi* iteration matrix for A is $k\sqrt{2}$. [6%]

(ii) For what values of k does Jacobi iteration converge? What is the rate of convergence? [3%]

(iii) For $k = 0.5$, how many iterations are needed for Jacobi iteration (used to approximate the solution to $A\mathbf{x} = \mathbf{b}$) to reach an accuracy of 10^{-5}? [5%]

b. For a general linear system $A\mathbf{x} = \mathbf{b}$, where A is $n \times n$, and a general splitting $A = M - N$, show that the iterative method $M\mathbf{x}^{(k+1)} = N\mathbf{x}^{(k)} + \mathbf{b}$ converges if the spectral radius of the matrix $M^{-1}N$ is less than 1. [8%]

c. If the iteration matrix in part ***b*** is strictly diagonally dominant, *must* the corresponding iterative method converge? Explain your answer. [3%]

I-2 **a.** Let $A = \begin{bmatrix} 6 & 4 & 2 \\ 3 & -2 & -1 \\ 3 & 4 & 1 \end{bmatrix}$.

Find the LDU decomposition of A, A = LDU, where L is unit lower-triangular, D is diagonal, and U is upper-triangular. [8%]

b. Let B be an $n \times n$ matrix. Show that if B can be factored as B = LU, where L is unit lower-triangular and U is upper-triangular, then this factorization is unique. [8%]

c. Given the linear system $C\mathbf{x} = \mathbf{b}$, where A has been factored as C= LU, and

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix},$$

solve $C\mathbf{x} = \mathbf{b}$ by forward/backward substitution (*without multiplying* L and U). [6%]

d. Briefly explain the role of "partial pivoting" in the Gaussian elimination procedure: What is it and what is its purpose? [3%]

I-3 The "Power Method" and the "QR Method" are techniques for finding approximations to the eigenvalues of a square matrix A.

a. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{x}^{(0)} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Then, A has eigenvalues 1 and 3 and corresponding eigenvectors $[1 \;\; 0]^T$ and $[1 \;\; 1]^T$, respectively.

(i) Apply *two* iterations of the Power Method to the matrix A with initial vector $\mathbf{x}^{(0)}$ to obtain $\mathbf{x}^{(2)}$, an approximation to the eigenvector of A corresponding to its dominant eigenvalue. [6%]

(ii) Will the Power Method converge to the dominant eigenvalue of A in this case? Explain why or why not. [4%]

b. **(i)** Obtain the QR factorization of the matrix A of part ***a***, and use this factorization to obtain the first iteration in the QR method. [9%]

(ii) Give one advantage and one disadvantage of the Power Method compared to the QR Method for finding the eigenvalues of an $n \times n$ matrix B. [6%]

PART II (Do two problems)

II-1 Consider the following elliptic boundary-value problem in the region $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$:

$$u_{xx} + u_{yy} = 0 \quad \text{(PDE)} \qquad 0 < x < 1, 0 < y < 1$$
$$u(0, y) = -y^2, \; u(1, y) = 1 - y^2, \quad 0 \le y \le 1$$
$$u(x, 0) = x^2, \; u(x, 1) = x^2 - 1, \quad 0 \le x \le 1$$

a. Show that $u(x, y) = x^2 - y^2$ is an exact solution of this boundary-value problem. [5%]

b. What are the maximum and minimum values achieved by the solution, u, to the given boundary-value problem in the region D? At what points (x,y) do they occur? [4%]

c. With $\Delta x = \Delta y = 1/3$, use the usual five-point difference scheme for approximating the given PDE to obtain a system of linear equations for solving this problem. Express this system in the form A**u** = **b**, where A is a 4×4 matrix. [12%]

d. Explain why the solution to your difference approximation in part ***c*** is unique. [4%]

II-2 a. Write a consistent explicit difference scheme for approximating the partial differential equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} + (x^2 + 1)\frac{\partial u}{\partial x}$$

Solve your scheme for $U_{i,j+1}$ [$= U(i\Lambda x, (j+1)\Lambda t)$], using $r = k/h^2$, but *do not show* that it is consistent. [5%]

b. Show that the simple (classical) explicit scheme for approximating $u_t = u_{xx}$ is consistent. [8%]

c. Use the Fourier (von Neumann) method to show that the following scheme for approximating $u_t = u_{xx}$ is unstable for all values of $r = k/h^2$:

$$\frac{U_{i,j+1} - U_{i,j-1}}{2k} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} \qquad [12\%]$$

II-3 Consider the partial differential equation (PDE) $u_{xx} + x\,u_{xy} - 2x^2 u_{yy} = 0$ with initial data given on the line $y = 0$.

a. Give the condition for determining whether a PDE of the form

$$A(x,y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} = 0$$

is elliptic, parabolic, or hyperbolic at the point (x,y), and use this condition to determine all values of x for which the given PDE is *hyperbolic*. [6%]

b. Determine the two characteristic *directions* (slopes of the characteristic curves, dy/dx) for the given PDE. [5%]

c. Find the *exact values* of the coordinates of the point of intersection, R, of the characteristic curves through the points P(1,0) and Q(2,0). [8%]

d. Write a scheme with $h = k$ $(\Delta x = \Delta y)$ that is consistent with the given PDE. (*Simplify* your scheme, but do *not* show that it is consistent.) [6%]