Department of Mathematics
California State University, Los Angeles
Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
FALL 2005

Do exactly 2 problems from part I AND exactly two problems from part II.

Part I: (Do two problems)

1. (a) Suppose that an n x n matrix is symmetric positive definite. Prove that:
   (i) A is non-singular.
   (ii) The diagonal entries of A satisfy $a_{kk} > 0$, for k = 1, 2, 3, …, n.

   (b) Let $B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$

   (i) Is B *positive definite*? (Use definition: A real symmetric matrix A is positive definite if $x^T Ax > 0$ for all nonzero x)
   (ii) Decompose B into the form $B = R^T R$, where R is an upper triangular matrix with positive diagonal entries.

2. (a) Factor $A = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$ into A = QR, where Q is an orthogonal matrix and R is an upper-triangular matrix.

   (b) Let B be an n x n matrix. Briefly describe the QR algorithm for finding the eigenvalues of B.

   (c) Let B_k be the kth iterate of the QR algorithm, Show that B_k is similar to B.

   (d) Let $x^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find $x^{(1)}$ and $x^{(2)}$ using the Power method for approximating the dominant eigenvalue of the matrix A in part (a).

3. (a) By computing the eigenvalues of the iteration matrix, determine whether or not the Gauss-Seidel method converges when applied to the system **B*x* = c**, where

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and c is arbitrary}$$

(b) Let A be an n x n strictly diagonally dominant matrix. Prove that the Jacobi iteration scheme for solving **Ax = b** converges for arbitrary b and every initial vector $x^{(0)}$.

(c) Give an advantage of using an iterative method (such as the above two) over Gaussian elimination in solving certain systems of linear equations.

Part II: (Do two problems)

1. Consider the problem for

$$u_t = 3u_{xx,} \qquad 0 \le x \le 1,\ t > 0$$
$$u(x,0) = f(x) \qquad 0 \le x \le 1,\ f(x)\ given$$
$$u(0,t) = u(1,t) = 0 \qquad t > 0$$

Suppose we approximate the PDE by the finite difference equation

$$\frac{u_{i,j+1} - u_{i,j}}{k} = 3\left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right], \text{ where } u_{i,j} = u(ih,\ jk), \qquad h = \Delta x,\ k = \Delta t$$

(a) Show that the finite difference equation is *consistent* with $u_t = 3u_{xx}$

(b) Let $h = \frac{1}{5}$, $r = \frac{k}{h^2}$. Find the 4x4 matrix A such that $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_{j+1} = A \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_j$, $\quad j = 0, 1, 2...$

(c) For what values of $r = \frac{k}{h^2}$ is the scheme stable? Explain.

2. (a) Show that for any x_0, and any $h > 0$, $f''(x_0)$ exactly equals

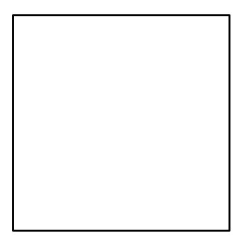
$$\frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2}$$

for any polynomial of degree 3 or less.

(b) Consider the problem $u_{xx} + u_{yy} = 0$ inside the square,

$$u(x,0) = x^2 \qquad 0 \le x \le 1$$
$$u(x,1) = x^2 - 1, \qquad 0 \le x \le 1$$
$$u(0,y) = -y^2 \qquad 0 \le y \le 1$$
$$u(1,y) = 1 - y^2 \qquad 0 \le y \le 1$$

$h = \frac{1}{3}$



(i)Verify that $u(x,y) = x^2 - y^2$ is a solution to the continuous problem.

(ii) Prove that $u(x,y) = x^2 - y^2$ is *the unique* solution to the problem. (you may assume the minimum and maximum principle for the continuous solution to $u_{xx} + u_{yy} = 0$)

(iii)Let $h = \frac{1}{3}$. Use the usual 5-point finite difference approximation to $u_{xx} + u_{yy} = 0$

and write down in matrix form the 4 x 4 system $Au_n = b$ that gives the finite difference solution $u_n = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ to the problem.

3. (a) Given $c^2 u_{xx} = u_{tt}$, $t > 0$; $u(x,0) = f(x)$, $\dfrac{\partial u}{\partial t}(x,0) = g(x)$

(i) Verify that the above PDE has a general solution of the form

$$u(x,t) = F(x+ct) + G(x-ct)$$

where *F* and *G* have continuous first and second partials.

(ii) Using the usual central difference approximation to approximate the partials in the above PDE, find the resulting finite difference equation in terms of r, $r = \frac{\Delta t}{\Delta x}$.(Simplify your answer for U^{j+1})

(iii) Find the characteristic curves and the interval of dependence for a point P in the domain.

(iv) State Courant-Friedrichs-Lewy Condition for convergence of the numerical solution to the exact solution.

(b) Suppose $c = 1, r = 1$;

(i) write down the finite-difference equation from (a) above.

(ii) Show that if the forward-difference condition is used to approximate the initial derivative condition, then

$$\left|e_i^1\right| \le \frac{1}{2} h^2 M$$

where $e = u - U$, $\Delta t = \Delta x = h$, and M is a constant. (Assume all partials of the exact solution are bounded in the solution domain)