Department of Mathematics
California State University, Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
FALL 2004

Instructions: Do any **2** problems from Part **I** AND any **2** problems from Part **II**

PART I (Do two problems)

I-1 **a**. In solving the linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is an arbitrary 3-vector and

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix},$$

determine whether or not the Jacobi iteration method converges. [9%]

b. Given the linear system $B\mathbf{x} = \mathbf{b}$, where B is a strictly diagonally dominant $n \times n$ matrix (and $\mathbf{b}$ is an arbitrary n-vector), prove that the Jacobi iteration matrix for this system, B_J, satisfies $\| B_J \|_\infty \le 1$. [8%]

c. Suppose that C is an $n \times n$ matrix that satisfies $\|C\| < 1$ and that $\mathbf{x}$ is an n-vector that satisfies $\mathbf{x} = C\mathbf{x} + \mathbf{c}$, where $\mathbf{c}$ is an arbitrary n-vector. Prove that the sequence defined by

$$x^{(k)} = C\mathbf{x}^{(k-1)} + \mathbf{c} \quad (k = 1, 2, 3, \ldots; \mathbf{x}^{(0)} \text{ arbitrary})$$

converges to $\mathbf{x}$ (that is, $\|\mathbf{x}^{(k)} - \mathbf{x}\| \to 0$) as $k \to \infty$. [8%]

I-2 **a.** Consider the following nonsingular matrix:

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{bmatrix}.$$

(i) Show that A cannot be directly factored as $A = LU$ (that is, show that this equation has no solution), where L is unit lower-triangular and U is upper-triangular. [9%]

(ii) Apply Gaussian elimination with row interchanges on A to obtain a permutation matrix P, a unit lower-triangular matrix L, and an upper-triangular matrix U such that $PA = LU$. [9%]

b. *Partial-pivoting* is a technique that is commonly used in conjunction with the Gaussian elimination method when solving a system of linear equations.

(i) Briefly explain what is meant by "partial-pivoting." [4%]

(ii) Briefly explain *the purpose* of using partial-pivoting in solving large linear systems by Gaussian elimination. [3%]

I-3 **a.** The "Power Method" and the "QR Method" are techniques for finding approximations to the eigenvalues of a square matrix A.

(i) Give sufficient conditions for the convergence of the Power Method. [4%]

(ii) Apply *two* iterations of the Power Method on the matrix

$$A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \quad \text{with initial vector} \quad \mathbf{x}^{(0)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

to obtain $\mathbf{x}^{(2)}$, an approximation to the eigenvector of A corresponding to its dominant eigenvalue. [6%]

(iii) Give one advantage of the Power Method over the QR Method. [3%]

b. Find the 3×3 matrix B that has eigenvalues $\lambda_1 = 4$, $\lambda_2 = 3$, $\lambda_3 = 2$ and corresponding orthogonal eigenvectors:

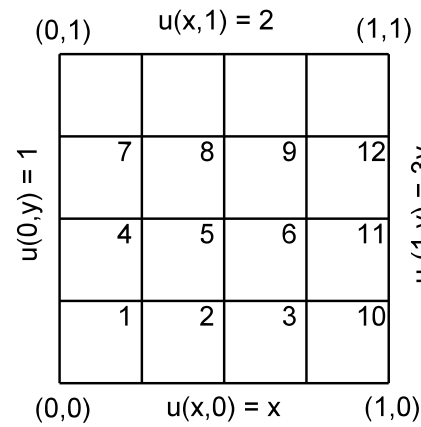
$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}. \qquad [12\%]$$

PART II (Do two problems)

II-1 Suppose that the function $u(x, y)$ satisfies

$$x u_{xx} + y u_{yy} = 3$$

inside the square region shown at the right (with $h = \Delta x = \Delta y = 1/4$), and that it satisfies the indicated boundary conditions.



a. Show that the given partial differential equation is elliptic in this region. [4%]

b. Denoting the finite difference solution at grid point *k* by u_k, ($k = 1, 2, ..., 12$), write finite difference equations consistent with the given PDE at grid points 4 and 5. (You do not have to show consistency.) [8%]

c. For a function $v(x, y)$, find constants c_0, c_1, and c_2 such that

$$\left| \frac{\partial v}{\partial x}(x_0, y_0) - [c_0 v(x_0, y_0) + c_1 v(x_0 - h, y_0) + c_2 v(x_0 - 2h, y_0)] \right| < K h^2$$

where K is a constant independent of h. (You may assume that $v(x, y)$ has as many partial derivatives as you need.) [8%]

d. Write an $O(h^2)$ finite difference equation to replace the boundary condition $u_x(1, \frac{1}{2}) = 3y$ at grid point 11. [5%]

II-2 Consider the initial-boundary value problem:

$$\begin{cases} u_t = u_{xx} & 0 \le x \le 1,\ t > 0 \\ u(x,0) = x & 0 \le x \le 1 \\ u(0,t) = 0, \quad u(1,t) = 1 & t > 0 \end{cases}$$

Suppose we approximate the PDE $u_t = u_{xx}$ by the finite difference scheme

$$-\lambda\theta U_{i-1,j+1} + (1+2\lambda\theta)U_{i,j+1} - \lambda\theta U_{i+1,j+1}$$
$$= \lambda(1-\theta)U_{i-1,j} + (1-2\lambda(1-\theta))U_{i,j} + \lambda(1-\theta)U_{i+1,j}$$

where $U_{i,j} = U(ih, jk)$, $\lambda = k/h^2$, and θ is a parameter with $0 \le \theta \le 1$.

a. For which value(s) of θ is this scheme *implicit*? [3%]

b. Describe (in one sentence each) what it means when we say that: [6%]

i. The given scheme is *consistent* with the given PDE.
ii. The given scheme is a *stable* approximation to the given PDE.

c. Let $\theta = 0$. For what values of λ is the resulting scheme convergent? (No proof is necessary.) [3%]

d. Let $\theta = 1$. For what values of λ is the resulting scheme convergent? (No proof is necessary.) [3%]

e. Taking $h = 1/3$, $k = 1/6$ (so $\lambda = 3/2$), and $\theta = 2/3$, give the system of two algebraic equations that relates $U_{1,j+1}$ and $U_{2,j+1}$ to $U_{1,j}$ and $U_{2,j}$. [10%]

II-3 **a.** Given the hyperbolic partial differential equation
$$u_{tt} = u_{xx} \quad (-\infty < x < \infty,\ t > 0),$$
show that the change of variables to the characteristic coordinates $\xi = x + t$, $\eta = x - t$ transforms this PDE into the form
$$u_{\xi\eta}(\xi,\eta) = 0 \quad (-\infty < \eta < \xi < \infty).$$ [10%]

b. Suppose we use the usual explicit scheme with $r = k/h = 1$,

$$U_{i,j+1} = U_{i+1,j} + U_{i-1,j} - U_{i,j-1},$$

together with forward differences to approximate the initial-value problem

$$\begin{cases} u_{tt} = u_{xx} & -\infty < x < \infty,\quad t > 0 \\ u(x,0) = f(x) & -\infty < x < \infty \\ u_t(x,0) = g(x) & -\infty < x < \infty \end{cases}.$$

Show that the numerical solution converges to the solution of the initial-value problem as $h \to 0$. [15%]