Department of Mathematics
California State University, Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
FALL 2003

Instructions: Do any **2** problems from Part **I** AND any **2** problems from Part **II**

PART I (Do two problems)

I-1 For $A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & 2 & 3 \\ 4 & 5 & 6 & 5 \end{bmatrix}$:

a. Find a basis for the row space of A.

b. Find a basis for the column space of A.

c. Find a basis for the nullspace of A.

d. Find a basis for the left nullspace of A ($= \{\mathbf{x} \mid \mathbf{x}^T A = \mathbf{0}^T\}$).

e. Are the first three columns of A linearly independent? Explain your answer.

f. Give an example of a vector $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has *no* solution.

g. Give an example of a vector $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ *has* a solution. Is the solution unique? Explain.

I-2 **a.** Let $B = \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}$, where β is a real number.

Show that Jacobi iteration converges for the linear system $B\mathbf{x} = \mathbf{b}$ (where **b** is arbitrary) if and only if B is strictly diagonally dominant.

b. Determine the set of all β for which the matrix B of part **a** is positive definite.

c. Show that Gauss-Seidel iteration converges in solving the linear system $U\mathbf{x} = \mathbf{b}$ (where **b** is arbitrary) for every nonsingular upper triangular matrix U.

I-3 Let $C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

a. Factor $C = LU$, where L is lower triangular with ones on the diagonal and U is upper triangular.

b. Use the result of part ***a*** to solve $Cx = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

c. Use the result of part ***a*** to factor $C = LDL^T$, where L is lower triangular with ones on the diagonal and D is diagonal.

d. Factor $C = QR$, where Q is an orthogonal matrix and R is upper triangular.

e. Briefly describe the Power Method for finding the eigenvalue of C of maximum modulus and its corresponding eigenvector.

f. Would the Power Method converge for the given matrix C if the initial vector were $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$? Why or why not?

PART II (Do two problems)

II-1 **a.** Derive the local truncation error for the usual 5-point difference scheme used to approximate the partial differential equation $u_{xx} + u_{yy} = 0$.

b. Given the boundary-value problem

$$u_{xx} + u_{yy} = x + y, \quad 0 < x < 1,\ 0 < y < 1$$
$$u(0, y) = u(1, y) = 0, \quad 0 \le y \le 1$$
$$u(x, 0) = u(x, 1) = \sin(\pi x), \quad 0 \le x \le 1$$

Determine the system of linear algebraic equation that results from solving this boundary-value problem using the 5-point scheme with $\Delta x = \Delta y = 1/3$. Simplify your system, writing it in the form $A\mathbf{x} = \mathbf{b}$.

II-2 **a.** Suppose we approximate the PDE $u_t = u_{xx}$ $(0 < x < 1, t > 0)$ by the finite difference scheme

$$-\lambda\alpha u_{i-1,j+1} + (1 + 2\lambda\alpha)u_{i,j+1} - \lambda\alpha u_{i+1,j+1}$$
$$= \lambda(1-\alpha)u_{i-1,j} + (1 - 2\lambda(1-\alpha))u_{i,j} + \lambda(1-\alpha)u_{i+1,j}$$

where $u_{i,j} = u(ih, jk)$, $\lambda = k/h^2$, and α is a parameter with $0 \le \alpha \le 1$. Show that this scheme is stable for all $\lambda > 0$ if $\frac{1}{2} \le \alpha \le 1$.

b. Approximate the initial-boundary-value problem

$$\begin{cases} u_t = u_{xx} & 0 < x < 1,\ t > 0 \\ u(x,0) = x(1-x), & 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0, & t \ge 0 \end{cases}$$

by the scheme in part **a** with $\alpha = 0$. Taking $h = \frac{1}{4}$ and $\lambda = \frac{1}{2}$, find the values of $u_{1,1}$, $u_{2.1}$, and $u_{3,1}$.

II-3 **a.** Consider the first-order partial differential equation

$$a u_x + b u_y = c,$$

where a, b, and c are functions of x, y, and u, and $u = u(x, y)$ is given on some initial curve Γ. Derive the equations satisfied by the characteristic curves.

b. Suppose that:

$$\begin{cases} u_x + x^2 u_y = 10x^2, & y > 0, \quad -\infty < x < \infty \\ u(x,0) = 3x^3, & -\infty < x < \infty \end{cases}$$

Calculate the value of y so that the point Q(2, y) is on the characteristic curve through the point P(1, 0).

c. Calculate the exact solution at Q(2, y), the point found in part **b**.

d. Use a numerical integration technique to calculate first approximations to the values of y_Q and u_Q.