Department of Mathematics
California State University, Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
FALL 2001

Instructions: Do any **2** problems from Part **I** AND any **2** problems from Part **II**

PART I (Do two problems)

I-1 Consider the following nonsingular matrix:

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{bmatrix}.$$

a. **(i)** Show that A cannot be directly factored as $A = LU$ (that is, show that this equation has no solution), where L is unit lower-triangular and U is upper-triangular.

(ii) Apply Gaussian elimination with row interchanges on A to obtain a permutation matrix P such that $PA = LU$, where L is unit lower-triangular and U is upper-triangular. (Give the matrices P, L, and U.)

b. Show that if B is an arbitrary nonsingular matrix such that $B = LU$, where L is unit lower-triangular and U is upper-triangular, then this factorization is unique; that is, if L_0 and U_0 are such matrices with $B = L_0U_0$, then $L = L_0$ and $U = U_0$.

I-2 Let $A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$

a. Find a basis for the row space of A.

b. Find a basis for the column space of A.

c. Find a basis for the nullspace of A $(= \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\})$.

d. Find a basis for the left nullspace of A $(= \{\mathbf{x} \mid \mathbf{x}^T A = \mathbf{0}^T\})$.

e. Are the first three columns of A linearly independent? Explain.

f. *Use the basis vectors of the left nullspace*, found in part **d**, to show that the system

$A\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$ has a solution.

g. Does the system $A\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$ have a unique solution? Explain.

I-3 **a.** By computing the eigenvalues of the iteration matrix, determine whether or not the Gauss-Seidel method converges when applied to the system $B\mathbf{x} = \mathbf{c}$, where

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{c} \text{ is arbitrary.}$$

b. Let A be an $n \times n$ strictly diagonally dominant matrix; that is:

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad (i = 1, 2, \ldots, n)$$

Prove that the Jacobi iteration scheme for solving $A\mathbf{x} = \mathbf{b}$, converges for arbitrary $\mathbf{b}$ and every initial vector $\mathbf{x}^{(0)}$.

c. Give an advantage of using an iterative method (such as Gauss-Seidel or Jacobi) instead of Gaussian elimination in solving certain systems of linear equations.

PART II (Do two problems)

II-1 Consider the following boundary-value problem:

$$u_{xx} + u_{yy} = 0$$
$$u(0, y) = -y^2,\ u(1, y) = 1 - y^2, \quad 0 \le y \le 1$$
$$u(x, 0) = x^2,\ u(x, 1) = x^2 - 1, \quad 0 \le x \le 1$$

a. Show that $u(x, y) = x^2 - y^2$ is the exact solution of this boundary-value problem..

b. What are the maximum and minimum values achieved by the solution, u, to the given boundary-value problem in the given region. At what points (x, y) do they occur?

c. With $\Delta x = \Delta y = 1/3$, use the usual five-point difference scheme for approximating the given PDE to obtain a system of linear equations for solving this problem. Simplify your system and express it in the form $A\mathbf{x} = \mathbf{b}$, where A is a 4×4 matrix.

II-2 a. Consider the first order PDE

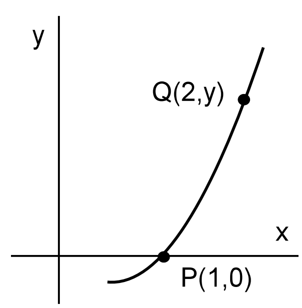
$$a u_x + b u_y = c$$

where a, b, and c are functions of x, y, and u. Suppose that $u = u(x, y)$ is given on some initial curve Γ. Derive the equations satisfied by the characteristic curves.

b. **(i)** Find the exact solution of

$$u_x + 3x^2 u_y = 1 + u \qquad y > 0,\ -\infty < x < \infty$$
$$u(x, 0) = x^2 \qquad -\infty < x < \infty$$

at the point $(2, y)$, where $(2, y)$ is on the characteristic curve through $(1, 0)$.



(ii) Use a numerical method for integrating along characteristic curves to find an approximate value for $(2, y)$, where $(2, y)$ is on the numerical characteristic through $(1, 0)$. (Just compute the first approximations, $y_Q^{(1)}$ and $u_Q^{(1)}$.)

II-3 Consider the following difference approximation to the PDE

$$u_t = a u_{xx} \qquad 0 < x < 1,\ t > 0 \qquad \text{(a is a constant, with a > 0)}$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = a\left[\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right],$$

where $u_{i,j} = u(ih, jk)$, $h > 0$, $k > 0$.

a. Is this an explicit or implicit scheme? Explain your answer.

b. Explain what it means to say that this scheme is *consistent* with the given PDE.

c. Explain what it means to say that this scheme *converges* to the given PDE.

d. Show that, for fixed k and h, this scheme is *stable* for all values of $r = k/h^2$.

e. Does this scheme *converge* to the given PDE? (Answer *yes* or *no*.)

f. Construct a finite difference approximation to the given PDE that is consistent, but is not stable. (You need not *show* that it is consistent and not stable!)