

**Department of Mathematics and Computer Science
California State University, Los Angeles**

Master's Degree Comprehensive Examination in

**NUMERICAL ANALYSIS
FALL 2000**

Instructions: Do any **2** problems from Part I AND any **2** problems from Part II

PART I (Do two problems)

- I-1 a.** Let A be a symmetric $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be the corresponding eigenvectors.
- (i) Suppose that the maximum of the $|\lambda_i|$ is less than 1 for $i = 1, 2, \dots, n$. Prove that for every vector \mathbf{v} , $\lim_{k \rightarrow \infty} A^k \mathbf{v} = \mathbf{0}$.
 - (ii) Suppose that there is one eigenvalue, say λ_j , for which $|\lambda_j| > 1$. Is it possible that $\lim_{k \rightarrow \infty} A^k \mathbf{v} = \mathbf{0}$ for some nonzero vector \mathbf{v} . Explain why or why not.
- b.** Let $B = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ and take $X^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- (i) Compute the first three iterates, $X^{(1)}, X^{(2)}, X^{(3)}$, using the Power Method for finding the eigenvalue of maximum modulus of the matrix B .
 - (ii) Give an example of an $X^{(0)}$ for which the Power Method (applied to the matrix B) will not yield the eigenvalue with largest modulus. Explain why it won't.

I-2 Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 2 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$

- Find a basis for the row space of A.
- Find a basis for the column space of A.
- Find a basis for the nullspace of A ($= \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$).
- Find a basis for the left nullspace of A ($= \{\mathbf{x} \mid \mathbf{x}^T A = \mathbf{0}^T\}$).
- Use the basis vectors of the left nullspace, found in part **d**, to answer the following questions:

(i) Does $A\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ have a solution? (ii) Does $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ have a solution?

I-3 a. Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$.

Find the LU decomposition of the matrix B; i.e., find a unit lower triangular matrix L and an upper triangular matrix U such that $B = LU$.

- b. Given the linear system $A\mathbf{x} = \mathbf{b}$, where A has been factored as $A = LU$, and

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} :$$

Solve $A\mathbf{x} = \mathbf{b}$ by forward/backward substitution (*without multiplying L and U*).

- Determine the LDU decomposition of the matrix A of part **b**; i.e., determine unit lower and upper triangular matrices L and U and diagonal matrix D such that $A = LDU$.
- Let C be an arbitrary nonsingular $n \times n$ matrix with *known* LU decomposition $C = LU$ and let \mathbf{d} be an arbitrary n -vector. Determine the number of *multiplications/divisions* it takes to solve $C\mathbf{x} = \mathbf{d}$ by forward/backward substitution.

PART II (Do two problems)

II-1 Consider Laplace’s equation in polar coordinates,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0,$$

for the region $\Omega = \{(x, y) \mid x^2 + y^2 < 1 \text{ and } y > 0\}$.

- a. Find a finite difference scheme with mesh widths of Δr and $\Delta \theta$ that approximates this equation.
- b. Suppose that the following boundary conditions are imposed on the given partial differential equation:

$$\begin{aligned} u(x, y) &= 100, & x^2 + y^2 &= 1, & y > 0 \\ u(x, y) &= 0, & y &= 0, & -1 < x < 1 \end{aligned}$$

Using your scheme of part **a** with $\Delta r = 0.5$ and $\Delta \theta = \pi/4$, obtain a system of linear equations that could be solved to approximate the solution to this boundary value problem at the grid points. *Simplify your system, but do not solve it.*

II-2 Given the initial-boundary value problem

$$\begin{cases} u_{tt} = 9u_{xx}, & 0 \leq x \leq 1, t > 0 \\ u(x, 0) = f(x), u_t(x, 0) = g(x), & 0 \leq x \leq 1 \quad (f(x), g(x) \text{ given}) \\ u(0, t) = u(1, t) = 0, & t > 0 \end{cases}$$

- a. Find the characteristic curves through the point $(0, 1/2)$.
- b. Derive an explicit finite difference scheme (taking $h = \Delta x$, $k = \Delta t$) for obtaining approximate solutions to this problem. Explain how to use this scheme to compute values along the “first row”; that is, when $t = k$.
- c. Show that your scheme of part **b** is consistent with the equation $u_{tt} = 9u_{xx}$.
- d. What values of $r = k/h$ will guarantee that the Courant-Friedrichs-Levy (CFL) condition will be satisfied for your scheme? If k/h is less than this value, what conclusion can you draw concerning the convergence of your scheme?

II-3 Consider the following difference approximation to

$$\begin{aligned}u_t &= c u_{xx} \quad (c > 0) & 0 < x < 1, \quad t > 0 \\u(x, 0) &= f(x) & 0 < x < 1 & \quad (f(x) \text{ given}) \\u(0, t) &= 0, \quad u(1, t) = 0 & t > 0\end{aligned}$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = c \left[\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2} \right], \text{ where } u_{i,j} = u(ih, jk)$$

- Show that this scheme is consistent with the given PDE.
- If this scheme is written as $B \mathbf{u}_{j+1} = C \mathbf{u}_j$, where $\mathbf{u}_j = (u_{1,j}, u_{2,j}, \dots, u_{N-1,j})$, determine the matrices B and C.
- Prove that this scheme is stable for all values of $r = k/h^2$.