Department of Mathematics
California State University, Los Angeles
Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
Spring 2007

Do exactly 2 problems from part I AND exactly two problems from part II.

Part I: (Do two problems)

1. Let A = $\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$.

   a. Give the value of $\|A\|_\infty$. [2]

   b. Is A *strictly* diagonally dominant (*yes* or *no*)? [2]

   c. Determine the LU decomposition of A. [8]

   d. Show that A is positive definite. [5]

   e. Let B_n denote the *n* x *n* matrix whose form is illustrated by $B_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

i.e., 1's on the diagonal, -1 in all below-diagonal positions, 1's in the last column, and all other entries 0. Show that if Gaussian elimination with partial pivoting is used, then B_n can be reduced to upper-triangular form without row exchanges, and the resulting matrix U has $u_{nn} = 2^{n-1}$ [8]

2.a. Let A be an *n* x *n* matrix with eigenvalues that satisfy $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$. If $\mathbf{x}^{(0)}$ is the initial vector for a Power Method iteration to find λ_1 , state sufficient conditions on $\mathbf{x}^{(0)}$ under which the Power Method converges to this eigenvalue. [4]

   b. Suppose that in applying the Power Method to the matrix A of part *a*, we obtain an approximation $\mathbf{x}^{(k)}$ to an eigenvector corresponding to λ_1. Describe how you could use $\mathbf{x}^{(k)}$ to find an approximation to λ_1. [3]

c. The matrix B = $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ has eigenvalues 1 and 2.

(i) Find eigenvectors corresponding to each of these eigenvalues. [5]
(ii) Use the result of part *c*(i) to give an initial vector $\mathbf{x}^{(0)}$ for which the Power Method will *not* converge to the eigenvalue 2. [2]

d. Find the QR factorization of the matrix B of part *c*. [8]

e. State one advantage of the QR Method over the Power Method in finding the eigenvalues of an *n* x *n* matrix A. [3]

3. Consider the system $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

a. Show that the Jacobi iteration scheme for this system converges. [8]

b. Find *k* such that $\left\|\mathbf{x}^{(k)} - \mathbf{x}\right\|_2 \le 10^{-3}\left\|\mathbf{x}^{(0)} - \mathbf{x}\right\|_2$, where $x^{(k)}$ is the k-th iterate of the Jacobi iteration . [4]

c. Is it possible to tell if Gauss-Seidel iteration scheme for this system converges from the result in *a*? Explain. [2]

d. Show that the Gauss-Seidel iteration scheme for this system converges. [8]

e. Find *k* such that $\left\|\mathbf{x}^{(k)} - \mathbf{x}\right\|_2 \le 10^{-3}\left\|\mathbf{x}^{(0)} - \mathbf{x}\right\|_2$, where $x^{(k)}$ is the k-th iterate of the Gauss-Seidel iteration. [3]

Part II: (Do two problems)

1. Suppose we approximate the PDE $U_t = U_{xx}$ by the finite-difference scheme

$$(u_{i,n+1} - u_{i,n-1}) / 2k = (u_{i-1,n} - 2u_{i,n} + u_{i+1,n}) / h^2,$$

where $k = \Delta t$, $h = \Delta x$, $r = k/h^2$, and $u_{i,n} = u(ih, nk)$.

a. Is the given scheme *explicit* or *implicit*? Why? [3]

b. If the given scheme is stable for a particular value of *r*, must it converge for that value of *r*? Explain why or why not. [4]

c. State Greschgorin's Circle Theorem (GCT). [4]

d. Apply GCT to establish stability condition ($0 < r \le \frac{1}{2}$) of the classical explicit scheme Approximating $U_t = U_{xx}$ with known boundary and initial values. [8]

e. Is Greschgorin's first theorem ($\rho(A) \le \|A\|_1$ or $\|A\|_\infty$) adequate for establishing the stability condition referred to in (d) above? Support your conclusion. [6]

2. Consider the initial value problem
$$2U_x + U_y = 1, \quad -\infty < x < \infty, \quad y > 0$$
$$U(x,0) = x^2, \quad -\infty < x < \infty$$

a. Find the characteristic curves for the problem above. [4]

b. Find the exact solution to the problem at (x, y) = (4, 1). [4]

c. Show that the finite difference scheme below is consistent with $2U_x + U_y = 0$

$$2(u_{i+1,j} - u_{i-1,j}) / 2h + (u_{i,j+1} - u_{i,j}) / k = 0, \text{ where } k = \Delta y,\ h = \Delta x$$ [9]

d. Show that the scheme in c is unstable for all values of r = k/h [8]

3. Consider the following elliptic boundary value problem:

$$U_{xx} + U_{yy} = x \qquad 0 < x < 1,\ 0 < y < 2,$$

$$U(x, 0) = x^2,\ U(x, 2) = (x - 2)^2 \qquad 0 < x < 1,$$

$$U(0, y) = y^2,\ U(1, y) = (y - 1)^2 \qquad 0 < y < 2.$$

a. Write a consistent 5-point finite difference scheme to the given PDE and *simplify it*. (You need not show that your scheme is consistent.) [5]

b. Using the scheme of part *a* with $\Delta x = \Delta y = 0.5$, determine the resulting system of 3 linear algebraic equations, A**u** = **b**. (*Simplify* these equations.) [8]

c. For *general* $\Delta x = \Delta y$ give the form of the n x n coefficient matrix A for the resulting system of linear equations. [6]

d. Briefly explain why A is invertible. [3]

e. Briefly explain the meaning of stability for any finite difference scheme. Explain whether or not it applies to your scheme in *a* above. [3]