Department of Mathematics
California State University, Los Angeles
Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
FALL 2006

Do exactly 2 problems from part I AND exactly two problems from part II.

Part I: (Do two problems)

1. **a.** Without multiplying the factors of $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ justify the following:

(i) The matrix A is non-singular. [4%]
(ii) The matrix A is symmetric. [4%]
(iii) The matrix A is positive definite. [4%]

b. What is this decomposition called? Are the matrices in this decomposition uniquely determined? [4%]

c. Count exact flops (all four arithmetic operations) required to obtain this decomposition of A. (show all steps) [6%]

d. Rewrite this decomposition in the form LDL^T where L is a lower triangular matrix and D is a diagonal matrix. [3%]

2. Let A be a 2x2 matrix with eigenvalues $\lambda_1 = 2, \lambda_2 = 1$ and corresponding eigenvectors $e_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a. Find A. [5%]

b. Suppose the power method were applied to the matrix A above with $x^{(0)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

i.e. $x^{(1)} = Ax^{(0)},\ x^{(2)} = Ax^{(1)}, \ldots$ etc.
Would the scheme converge (with scaling) to give the eigenvalue of A of largest magnitude and its corresponding eigenvector? Explain. [4%]

c. Apply Power method (2 iterations) to the matrix found in (a) to approximate the largest eigenvalue. Can this method also provide the other eigenvalue? Explain. [4%]

d. Apply QR algorithm (2 iterations) to above A to approximate the eigenvalues. [8%]

e. Give one advantage and one disadvantage of power method over QR method. [4%]

3. **a.** Let $B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$

(i) By computing the spectral radius of the iteration matrix, determine whether or not Jacobi iteration converges in solving $B\mathbf{x} = \mathbf{c}$, where $\mathbf{c}$ is an arbitrary 3-vector. [9%]

(ii) Without doing any further work can you determine whether the Gauss-Seidel iteration will converge for the above system or not? Explain. [2%]

(iii) Note that the above matrix *B* is positive definite and tri-diagonal. Based on these facts and your result in part (i) determine the spectral radius of the Gauss-Seidel iteration matrix. [3%]

b. Given the linear system $A\mathbf{x} = \mathbf{b}$ where *A* is an $n \times n$ matrix and $\mathbf{b}$ is an arbitrary n-vector, write $A = M - N$, where M is nonsingular, and consider the iterative method

$$\mathbf{x}^{(k+1)} = M^{-1}N\mathbf{x}^{(k)} + M^{-1}b \quad (k = 0, 1, 2, 3, \ldots)$$

(i) Show that the error in the above approximation satisfies:

$$\left\|\mathbf{x}^{(k)} - \mathbf{x}\right\| \le \|G\|^k \left\|\mathbf{x}^{(0)} - \mathbf{x}\right\|,$$

where $G = M^{-1}N$ and $\mathbf{x}^{(0)}$ is the initial approximation. [6%]

(ii) Let ρ be the spectral radius of the matrix $G = M^{-1}N$, and assume that $\rho < 1$. Use the result of part (i) to show that if m is a positive integer and $k \ge m/-\log_{10}\rho$, then $\left\|\mathbf{x}^{(k)} - \mathbf{x}\right\| \le C(10^{-m})$, where C is a constant that is independent of k and m. [5%]

Part II: (Do two problems)

1. **a.** Given: $\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[a(x)\frac{\partial u}{\partial x}\right] + ku \qquad a(x) \neq 0$

   Show that, in approximating u_t by the usual forward difference formula, and u_x, u_{xx} by the usual central differences, the resulting explicit finite-difference scheme for approximating the above PDE is *consistent*. [13 %]

   b. (i) Write down a consistent explicit finite-difference scheme for approximating the solution to:

$$\frac{\partial u}{\partial t} = (1+x^2)\frac{\partial^2 u}{\partial x^2} - u \qquad 0 < x < 1 \text{ and } t > 0$$
$$u(x,0) = x(1-x), \qquad 0 \le x \le 1$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0$$

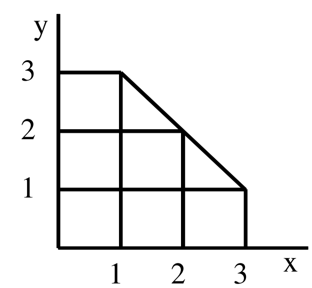
   Simplify your answer. [4%]

   (ii) Use $h = \Delta x = 0.2$ and $k = \Delta t = 0.01$ to compute an approximation to the exact solution at the given points:

   $u(0.2, 0.01), \quad u(0.4, 0.01)$

   Simplify your answers. Do not approximate any fractions. [8%]

2. a. Consider the PDE $U_{xx} + U_{yy} = x^2 + y^2$ defined on the region R (with the square mesh shown) at the right. Suppose that we approximate this PDE by the usual 5-point scheme.

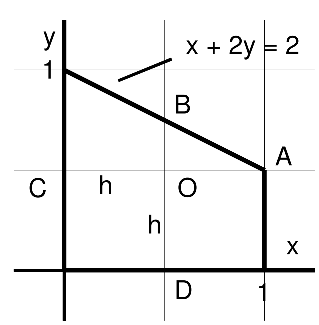


   i. For the boundary values $U(x, y) = x + y$, determine the matrix A and vector **b** of the linear system A**u** = **b** that results from this approximation. [8%]

   ii. Now, suppose that for $x = 0$ $(0 < y < 3)$, the boundary values are $\partial U/\partial n = 0$, and all other boundary values are given by $U = 2$.
   Approximating $\partial U/\partial n$ by central differences, express the approximate solution $u(0, 1)$ in terms of $u(0, 2)$ and $u(1, 1)$. [8%]

b.

Consider the PDE $U_{xx} + U_{yy} = 0$ in the trapezoidal region at the right ($h = 1/2$), with $U(x, y) = 2x$ on the boundary. Use the *weighted* 5-point approximation to find the value of u_O. [9%]

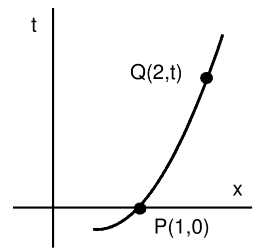


3. **a.** Given the first-order initial-value problem (IVP):

$$U_x + 3x^2 U_t = U \qquad t > 0, \ -\infty < x < \infty$$
$$U(x, 0) = x^2 \qquad -\infty < x < \infty$$

i. Find an *equation* for the characteristic curve through the point (1, 4). [5%]

ii. By writing and solving an equation for dU/dx, find the value of the *exact solution* of the given IVP at the point (2, t), where (2, t) is on the characteristic curve through (1, 0). [8%]



b. Consider the PDE:

$$U_{xx} + (1 - 2x)U_{xt} + (x^2 - x - 2)U_{tt} = 0$$

i. Determine all values of (x, t) for which this PDE is hyperbolic. [6%]

ii. Write a consistent approximation to U_{xt}. (You need not show that it is consistent.) [6%]