Department of Mathematics
California State University, Los Angeles

Master's Degree Comprehensive Examination

NUMERICAL ANALYSIS
SPRING 2006

Instructions: Do any **2** problems from Part **I** AND any **2** problems from Part **II**

PART I (Do two problems)

I-1 **a.** Let $A = \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$, where α, β, and γ are real numbers with $\alpha > 0$ and $\beta > 0$.

(i) Give necessary and sufficient conditions on α, β, and γ under which A is strictly diagonally dominant. [2%]

(ii) Find the eigenvalues (in terms of α, β, and γ) of the Jacobi iteration matrix associated with A. [6%]

(iii) Under what conditions on α, β, and γ is the spectral radius of the Jacobi iteration matrix for A less than 1? [3%]

b. Given the system of linear equations $B\mathbf{x} = \mathbf{b}$, where B is an $n \times n$ matrix, *prove* that if B is strictly diagonally dominant, then Jacobi iteration applied to this linear system converges for arbitrary **b**. [10%]

c. Give an example of a square matrix C that is positive definite, but not diagonally dominant. Justify your answer. [4%]

I-2 Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

a. Find bases for each of the four fundamental subspaces of A — its row space, column space, nullspace, and left nullspace. [12%]

b. What is the rank of A? [2%]

c. Explain why or why not there exists a vector **b** for which the linear system $A\mathbf{x} = \mathbf{b}$ fails to have a solution. [2%]

d. If possible, find a vector:

- **x** that is orthogonal to the row space of A;
- **y** that is orthogonal to the column space of A;
- **z** that is orthogonal to the nullspace of A.

(If it is not possible to find **x, y** or **z**, state this fact.) [6%]

e. Construct a matrix B whose column space contains the vectors (1, 2, 5) and (0, 3, 1), and whose nullspace contains the vector (1, 1, 2). [3%]

I-3 **a.** Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$.

(i) Factor A as $A = QR$, where Q is an orthogonal matrix and R is an upper-triangular matrix. [8%]

(ii) Let $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ using the Power Method for approximating the dominant eigenvalue of the matrix A. [6%]

b. Let B be an $n \times n$ matrix.

(i) Briefly describe the QR algorithm for finding the eigenvalues of B. [3%]

(ii) Let B_k be the k^{th} matrix iterate of the QR algorithm applied to B. Show that B_k is similar to B. [4%]

(iii) Give two different reasons why the Power Method applied to B might *not* converge to an eigenvector corresponding to the eigenvalue of B of largest modulus. [4%]

PART II (Do two problems)

II-1 Suppose that central differences (with $\Delta x = \Delta y$) are used to construct the usual 5-point scheme approximating the elliptic partial differential equation (PDE)

$$u_{xx} + u_{yy} = x$$

inside the rectangular region $\{(x, y) \mid 0 \le x \le 1, 0 \le y \le 2\}$ with boundary conditions:

$$u(x, 0) = x^2, \quad u(x, 2) = (x - 2)^2 \qquad (0 \le x \le 1)$$

$$u(0, y) = y^2, \quad u(1, y) = (y - 1)^2 \qquad (0 \le y \le 2)$$

- **a.** Give the usual 5-point difference scheme for approximating the given PDE. [4%]
- **b.** Find the (local) truncation error for this scheme. [9%]
- **c.** Find *and simplify* the linear system that results from applying this scheme to the given boundary value problem with $\Delta x = \Delta y = 1/2$. [9%]
- **d.** Explain why the algebraic linear system, $A\mathbf{u} = \mathbf{b}$, that results from solving this difference scheme for *arbitrary* $\Delta x = \Delta y$ has a unique solution. [3%]

II-2 Suppose we approximate the partial differential equation (PDE) $u_t = u_{xx}$ by the scheme

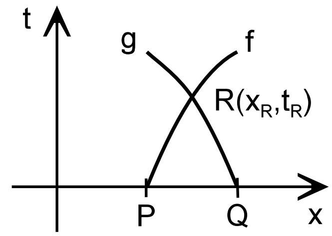
$$-(r/2)u_{i-1,j+1} + (1+r)u_{i,j+1} - (r/2)u_{i+1,j+1} = (r/2)u_{i-1,j} + (1-r)u_{i,j} + (r/2)u_{i+1,j}$$

where $u_{i,j} = u(ih, jk)$ and $r = k/h^2$.

- **a.** What is the name of this scheme? [2%]
- **b.** Is this an implicit scheme or an explicit scheme? [2%]
- **c.** Determine the matrices B and C so that this scheme takes the form $B\mathbf{u}_{j+1} = C\mathbf{u}_j$, where $\mathbf{u}_j = [u_{1,j} \ u_{2,j} \ \ldots \ u_{N-1,j}]^T$ (assuming zero boundary values). [6%]
- **d.** Show that this scheme is stable for all positive values of *r*. (If need be, you may assume that for the matrices B and C of part ***c***, B^{-1} exists and $B^{-1}C$ is symmetric.) [12%]
- **e.** If the given scheme is used to approximate an initial-boundary value problem with PDE $u_t = u_{xx}$, zero boundary values, and a given continuous initial function, explain why we can conclude from part ***d*** that the scheme *converges* for all positive *r*. [3%]

II-3 For the initial value problem (IVP):

$$u_{xx} + (1 - 2x)u_{xt} + (x^2 - x - 2)u_{tt} = 0 \qquad (-\infty \le x \le \infty,\ t > 0)$$
$$u(x, 0) = x \qquad (-\infty \le x \le \infty)$$
$$u_t(x, 0) = 1 \qquad (-\infty \le x \le \infty)$$



a. Show that the given PDE is hyperbolic for all (x,t). [5%]

b. Give the characteristic *directions* (slopes of the characteristic curves) of this PDE. [5%]

c. Using the method of characteristics, *set up the equations* whose solution would give an approximation to the solution of *this* IVP at the grid point R, the intersection of the characteristic curves f and g, through P(.4,0) and Q(.5,0). (Do *not* solve any of your equations.) [10%]

d. Give a consistent finite-difference approximation *to the term* $(1 - 2x)u_{xt}$. (You need *not* show that your approximation is consistent.) [5%]