Department of Mathematics
California State University, Los Angeles
Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
SPRING 2008

Do exactly 2 problems from part I AND 2 problems from part II.

Part I: (Do two problems)

1. Let $B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$.

(a) By computing the spectral radius of the iteration matrix, determine whether or not Jacobi iteration converges in solving $B\mathbf{x} = \mathbf{c}$ for an arbitrary 3-vector $\mathbf{c}$.

(b) Without doing any further work can you determine whether the Gauss-Seidel iteration will converge for solving $B\mathbf{x} = \mathbf{c}$ or not? Explain.

(c) Note that the above matrix B is positive definite and tridiagonal. Based on this and your results for part (a) determine the spectral radius of the Gauss-Seidel iteration matrix.

(d) Given the linear system $A\mathbf{x} = \mathbf{b}$ where A is an $n \times n$ matrix and $\mathbf{b}$ an n-vector, write $A = M - N$ where M is nonsigular and consider the interative scheme

$$\mathbf{x}^{(k+1)} = M^{-1}N\mathbf{x}^{(k)} + M^{-1}\mathbf{b} \quad (k = 1, 2, 3, \ldots).$$

Show that

$$||\mathbf{x}^{(k)} - \mathbf{x}|| \leq ||G||^k ||\mathbf{x}^{(0)} - \mathbf{x}||,$$

where $G = M^{-1}N$, $\mathbf{x}^{(0)}$ is the initial approximation, and $\mathbf{x}$ is the actual solution.

2. $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 5 & 7 & 2 \\ 2 & 7 & 14 & 3 \\ 1 & 2 & 3 & 3 \end{bmatrix}$.

(a) Find a decomposition of A in the form $A = R^T R$, where R is a upper triangular matrix.

(b) For a nonsingular matrix M show that $B = M^T M$ is positive definite.

(c) For a positive definite matrix $C = [c_{ij}]$ show that $c_{ii} > 0$.

(d) Show that the C in part (c) is nonsingular.

3.
(a) Given a $n \times n$ matrix A and with $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$, where λ_i is an eigenvalue of A,
(i) Describe the power method to find λ_1, and its corresponding eigenvector.
(ii) Show the convergence is linear.

(b) Describe briefly how the Rayleigh Quotient Iteration method improves the rate of convergence for the above A.

(c) Do two steps of the Rayleigh Quotient Iteration method on the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ and compare the theoretical and observed rates of convergence.

Part II: (Do two problems)

1. (a) Consider the first-order PDE

$$au_x + bu_y = c,$$

where a, b, and c are functions of x, y, and u and $u = u(x, y)$ is given on an initial curve Γ. Derive the equations satisfied by the characteristic curves.

(b) Suppose

$$\begin{aligned} &u_x + 2xu_y = x, \quad y > 0, \quad -\infty < x < \infty \\ &u(x, 0) = 4 \quad -\infty < x < \infty \end{aligned}$$

Calculate the value of y so that $Q(3, y)$ is on the characteristic curve through $P(2, 0)$.

(c) Compute the exact value u_Q, where Q is the point found in (b).

(d) Use the method of numerical characteristic to calculate first approximations to the value of y_Q and u_Q.

2. Consider the problem

$$\begin{aligned} &u_t = 3u_{xx} \quad 0 < x < 1, t > 0 \\ &u(x, 0) = f(x) \quad 0 \leq x \leq 1, f(x) \text{ given} \\ &u(0, t) = u(1, t) = 0 \quad t > 0. \end{aligned}$$

Suppose we approximate the PDE by the finite difference equation

$$\frac{u_{i,j+1} - u_{i,j}}{k} = 3\left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right]$$

where $u_{i,j} = u(ih, jk)$, $h = \Delta x$, and $k = \Delta t$.

(a) Show that the finite difference equation is consistent with $u_t = 3u_{xx}$.

(b) Let $h = 1/5$, $r = k/h^2$. Find the 4×4 matrix such that

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_{j+1} = A \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_j, \quad j = 0, 1, 2, \cdots$$

(c) For what values of $r = k/h^2$ is the scheme stable? Explain.

3. Consider the PDE

$$\begin{aligned}
&u_{xx} + u_{yy} = 0, \quad 0 < x < 1, 0 < y < 1 \\
&u(x,0) = x^2, \; u(x,1) = x^2 - 1 \quad 0 \le x \le 1 \\
&u(0,y) = -y^2, \; u(1,y) = 1 - y^2 \quad 0 \le y \le 1
\end{aligned}$$

(a) Show that $u(x,y) = x^2 - y^2$ is the exact solution to this problem.

(b) Find the maximum and minimum values of $u(x,y)$ and say at what point they occur.

(c) Using the standard 5-point difference scheme for approximating the PDE write out the equations you get for $\Delta x = \Delta y = 1/3$. Simplify them.

(d) For arbitrary Δx and Δy explain how you know the equations in part (c) have a unique solution.