

**Department of Mathematics**  
**California State University, Los Angeles**  
**Master's Degree Comprehensive Examination in**

**NUMERICAL ANALYSIS**  
**FALL 2008**

---

**Do exactly 2 problems from part I AND 2 problems from part II.**

---

**Part I: (Do two problems)**

1. Let  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ . Note that  $A$  is symmetric and nonsingular.

a. Is  $A$ :

i. Diagonally dominant?

ii. Diagonalizable?

iii. Positive definite?

iv. Orthogonal?

b. Give  $\|A\|_\infty$ .

c. Apply Gershgorin's Theorem to  $A$  to locate its eigenvalues; that is, to determine the interval in which all the eigenvalues lie.

d. Find the Jacobi iteration matrix and use it to determine whether or not Jacobi iteration converges for the given matrix  $A$ .

2. Let  $A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 4 \\ 2 & 4 & 6 & 6 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

a. Solve  $A\mathbf{x} = \mathbf{b}$ , using Gaussian elimination (without pivoting).

b. Find the row-reduced echelon form,  $R$ , of  $A$ .

c. Find a basis for the row space of  $A$ .

d. Is the column space of  $A$  equal to the column space of  $R$ ? (Answer yes or no.)

e. Find a vector  $\mathbf{b}_0$  such that  $A\mathbf{x} = \mathbf{b}_0$  has no solution.

- 3.** Let  $A = \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix}$  and  $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .
- a.** Using  $\mathbf{x}^{(0)}$  as an initial vector, do two iterations of the power method to find a vector  $\mathbf{x}^{(2)}$  that approximates the eigenvector associated with the eigenvalue of greatest magnitude.
  - b.** Give sufficient conditions for the convergence of the power method.
  - c.** Do one iteration of the QR method for finding eigenvalues using  $A$ .

**Part II: (Do two problems)**

1. Consider the boundary value problem:

$$\begin{aligned} u_{xx} + 2u_{xy} + 3u_{yy} &= 0 & 0 < x < 1, 0 < y < 1 \\ u(0, y) = u(1, y) &= y & 0 \leq y \leq 1 \\ u(x, 0) &= x(x - 1) & 0 \leq x \leq 1 \\ u(x, 1) &= 1 & 0 \leq x \leq 1 \end{aligned}$$

- a. What type of differential equation is in this problem (hyperbolic, parabolic, or elliptic)? Justify your answer.
- b. Write a finite difference scheme for the differential equation in this problem using the usually 5 point scheme for  $u_{xx} + u_{yy}$ . (Hint: Use forward differencing twice for  $u_{xy}$ .)
- c. Determine the system of equations that results from solving the boundary value problem using the scheme in **b** and  $\Delta x = \Delta y = 1/3$ . Simplify your answer.

2. Consider the following difference approximation to the P.D.E

$$\begin{aligned} u_t = u_{xx} & \quad 0 < x < 1, t > 0 \\ \frac{u_{i,j+1} - u_{i,j-1}}{2k} &= \frac{u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + u_{i+1,j}}{h^2} \end{aligned}$$

where  $u_{i,j} = u(ih, jk)$ ,  $h > 0, k > 0$ .

- a. Is this an explicit or implicit scheme? Justify your answer.
- b. Show that the right-hand side of the given scheme is a consistent approximation to  $u_{xx}$ .
- c. Use the von Neumann (Fourier) method to determine for which  $r = k/h^2$  this scheme is stable.

**Part II Continued: (Do two problems)**

3. Consider the initial-boundary value problem:

$$\begin{aligned}u_{tt} &= (1/4)u_{xx} + 3u_x + u & 0 < x < 3, t > 0 \\u(x, 0) &= x^2, \quad u_t(x, 0) = 3x & 0 \leq x \leq 3 \\u(0, t) &= 0 \quad u(3, t) = 0 & t > 0\end{aligned}$$

Suppose that we place a mesh (grid) on the region with spacing  $h = \Delta x$ ,  $k = \Delta t$ , and let  $u_{i,j} = u(i\Delta x, j\Delta t)$ .

- a. Derive an explicit scheme to obtain  $u_{i,j+1}$  from  $u_{i,j}$  and  $u_{i,j-1}$ .
- b. Letting  $\Delta x = \Delta t = 1$ , use the scheme of part **a**, together with the initial and boundary conditions, to find  $u_{1,1}$ , the approximation to  $u(1, 1)$ .
- c. Find the point of intersection, **R**, of the characteristic curves for the given P.D.E through the points  $P(1, 0)$  and  $Q(2, 0)$ .
- d. What restriction on the ration  $k/h$  is given by the slopes of the characteristic curves?