ALGEBRA COMPREHENSIVE EXAMINATION

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Directions: Answer 5 questions only. You must answer *at least one* from each of groups, rings, and fields. Be sure to show enough work that your answers are adequately supported.

Notation: If n is a positive integer, let $\mathbb{Z}_n$ denote the integers modulo n. Let $\mathbb{Q}$ denote the rational numbers.

Groups

1. Show that all groups of order 45 are abelian.
2. Let G be a cyclic group and H a subgroup of G. Prove that H is cyclic.
3. Let G be a finite group with $|G| > 1$, and let $\mathrm{Inn}(G)$ be the group of inner automorphisms of G. Show that if G is isomorphic to $\mathrm{Inn}(G)$, then $|G|$ has at least two distinct prime factors. (Hint: Do a proof by contradiction.)

Rings

1. Let p be a prime number. Let $D : \mathbb{Z}_p \to \mathbb{Z}_p$ be a function such that $D(a \cdot b) = a \cdot D(b) + b \cdot D(a)$ for all $a, b \in \mathbb{Z}_p$. Prove that D is the zero map.
2. Let D be a Euclidean domain and $a, b, c \in D$. Prove:
   (a) If a divides bc and $GCD(a, b) = 1$, then a divides c.
   (b) If a is irreducible, then a is prime.
3. Let R be a commutative ring with identity 1. Prove that an ideal M is maximal if and only if R/M is a field.

Fields

1. Let $\mathbb{Q}$ be the field of rationals and let $p(x) = x^3 - 4x + 5$. Assume α is a root of $p(x)$.
   (a) Prove that $p(x)$ is irreducible over $\mathbb{Q}$.
   (b) Find $a, b, c \in \mathbb{Q}$ such that $(\alpha + 1)^{-1} = a + b\alpha + c\alpha^2$.
2. Let F be a field. Let G be a finite subgroup of the group of units of F. Prove that G is cyclic. (Hint: Do a proof by contradiction. First show that G is a finite abelian group. To get a contradiction, find a positive integer n such that the polynomial $x^n - 1$ has more than n zeroes. You will need to use a major theorem about finite abelian groups.)
3. Let $\xi = e^{2\pi i/n}$ be a primitive n-th root of unity. Prove that $Gal(\mathbb{Q}(\xi)/\mathbb{Q}) \cong \mathbb{Z}_n^\times$. Note: $\mathbb{Z}_n^\times$ is the group of units under multiplication in $\mathbb{Z}_n$.