ALGEBRA COMPREHENSIVE EXAMINATION

Fall 2008

Brookfield*, Chabot, Shaheen

Directions: Answer 5 questions only. You must answer *at least one* from each of groups, rings, and fields. Be sure to show enough work that your answers are adequately supported.

Groups

(1) Show that any group of order 15 is cyclic.

(2) Let N and H be subgroups of a group G with N normal. Show that $NH = \{nh \mid n \in N \text{ and } h \in H\}$ is a subgroup of G.

(3) Let p be a prime number and G a nontrivial finite p-group with center $Z(G)$.
   (a) Show that $Z(G)$ is nontrivial.
   (b) Let N be a nontrivial normal subgroup of G. Show that $N \cap Z(G)$ is nontrivial.

Rings

(1) Let R be a finite commutative ring (not necessarily with a multiplicative identity) with more than one element and no zero divisors.
   (a) Show that R has a multiplicative identity and so is a domain.
   (b) Show that R is a field.

(2) Let R be the set of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ with $a, b \in \mathbb{R}$ together with the usual matrix addition and multiplication operations. Show that R is isomorphic to $\mathbb{C}$.

(3) Let R be a commutative ring with identity and M an ideal of R. Show that M is maximal if and only if, for every $r \in R \setminus M$, there is an $x \in R$ such that $1 - rx \in M$. Note: $R \setminus M = \{r \in R \mid r \notin M\}$.

Fields

(1) Let E be the splitting field of $x^6 - 3$ over the rational numbers $\mathbb{Q}$.
   (a) Find $[E : \mathbb{Q}]$. Explain.
   (b) Show that the Galois group $Gal(E/\mathbb{Q})$ is not abelian.

(2) Let E be an extension field of F with $[E : F] = 5$.
   (a) Show that $F(\alpha) = F(\alpha^3)$ for all $\alpha \in E$.
   (b) Show that $F(\alpha) = F(\alpha^9)$ for all $\alpha \in E$.

(3) Let K be the splitting field of $f(x) = x^3 + 3x^2 + 3x + 2 \in \mathbb{Z}_5[x]$ over $\mathbb{Z}_5$.
   (a) Is f irreducible over $\mathbb{Z}_5$?
   (b) How many elements does K have?
   (c) Factor f completely in $K[x]$.