California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Real and Functional Analysis   Spring 2009

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Do **five** of the following problems. As follows.
The problems are divided into three groups:

A Advanced calculus and classical analysis
B1 Measure and integration
B2 Functional analysis

Select problems as follows:
(1) Select at least two problems from part **A** (1-5).
(2) Select at least two problems from either part **B1** (6-9) or part **B2** (10-14).
(3) Select any fifth problem.

Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation:
R denotes the set of real numbers.
N denotes the set of positive integers.
$|z|$ denotes the absolute value of the number z.
$\{x_n\}_{n=1}^{\infty}$ denotes a sequence $x_1, x_2, x_3, \ldots$.
If A and B are sets, then $A \setminus B$ denotes the set difference $A \setminus B = \{x \in A : x \notin B\}$.

Part A: Advanced Calculus and Classical Analysis

Spring 2009 # 1. Compute each of the following

(a) $F'(x)$ where $F(x) = \int_{\sqrt{x}}^{17} e^{t^2}\,dt$ where $x > 0$;

(b) $\int_{e}^{8e} \frac{1}{x(\ln x)^2}\,dx$

(c) $\int_{0}^{4} x\sqrt{2x+1}\,dx$

Spring 2009 # 2. Suppose $f : \boldsymbol{R} \to \boldsymbol{R}$ is a differentiable function such that $f(x)$ and $f'(x)$ are never both 0 at the same x. Show that $f(x) = 0$ can have at most finitely many solutions in the closed interval [0, 1]. (Hint: Use Rolle's Theorem)

Spring 2009 # 3. State and prove the product rule of differentiation.

Spring 2009 # 4. Let $\{x_n\}_{n=1}^{\infty}$ be a sequences of real numbers and let $p \in \boldsymbol{R}$. Prove the following are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ has a subsequence convergent to p;

(b) $\forall \varepsilon > 0 \forall n \in \boldsymbol{N}, \exists\, j > n$ with $|x_j - p| < \varepsilon$;

(c) $p \in \bigcap_{k=1}^{\infty} \mathrm{cl}\,(\{x_n : n \geq k\})$.

Spring 2009 # 5. Find the Maclaurin series, i.e., the Taylor series expanded about zero, for $f(x) = e^{2x}$. Prove that for each $a > 0$, the series converges uniformly to f on $[-a, a]$.

Part B1: Measure and Integration

Spring 2009 # 6. (a) Let Ω be a set and let Σ be a set of subsets of Ω. Define what it means for Σ to be a σ-algebra of subsets of Ω.

(b) Let (Ω, Σ, μ) be a measure space. Let Σ_0 be a subset of Σ such that $A \in \Sigma_0$ if and only if either $\mu(A) = 0$ or $\mu(A^c) = 0$. Is Σ_0 an algebra of subsets of Ω? Is Σ_0 a σ-algebra of subsets of Ω? Justify your answers.

Spring 2009 # 7. (a) Let (Ω, Σ) be a measurable space. Define what it means for a function $f: \Omega \to \boldsymbol{R}$ to be Σ-measurable.

(b) Let (Ω, Σ, μ) be a finite measure space, and let $f: \Omega \to \boldsymbol{R}$ be a measurable function. For each integer $n \geq 1$, let $A_n = \{x \in \Omega : f(x) > n\}$. Prove that $\lim_{n\to\infty} \mu(A_n) = 0$.

Spring 2009 # 8. Let (Ω, Σ, μ) be a measure space.

(a) What does it mean for a sequence of measurable functions f_n to converge in measure to a function f?

(b) Let (Ω, Σ, μ) be a finite measure space, and let $\{f_n, n \geq 1\}$ be a sequence of Σ-measurable functions. For every $n \geq 1$, let $g_n = (f_n + f_{n+1})/2$. Use the definition of convergence in measure to prove that if f_n converges to f in measure, then the sequence g_n converges to f in measure.

Spring 2009 # 9. Let (Ω, Σ, μ) be a finite measure space, and let $f: \Omega \to \mathbf{R}$ be a Lebesgue integrable function. Show that if $\int_E f d\mu = 0$ for every $E \in \Sigma$, then $f = 0$ a.e.

Part B2: Functional Analysis

Spring 2009 # 10. Let X and Y be Banach spaces and let $T: X \to Y$ be linear. Prove that the following conditions are equivalent:

(a) T is continuous at the origin 0 of X;

(b) $\sup \{\|T(x)\| : \|x\| \leq 1\} < \infty$;

(c) T is Lipschitz continuous: $\exists\ M > 0$ such that $\forall x_1, x_2$ in X, $\|Tx_1 - Tx_2\| \leq M\|x_1 - x_2\|$.

Spring 2009 #11. Prove the Uniform Boundedness Principle: if $\boldsymbol{T}$ is a family of continuous linear transformations from a Banach space X to a Banach space Y such that at each $x \in X$, $\{T(x) : T \in \boldsymbol{T}\}$ is bounded in Y, then the family of transformations $\boldsymbol{T}$ is uniformly bounded with respect to the operator norm.

Spring 2009 #12. Let $C_b(\boldsymbol{R})$ be the vector space of bounded real valued functions on $\boldsymbol{R}$. Define a norm on $C_b(\boldsymbol{R})$ by $\|f\|_\infty := \sup \{|f(x)| : x \in \boldsymbol{R}\}$. Prove that this is indeed a norm that makes $C_b(\boldsymbol{R})$ a Banach space.

Spring 2009 #13. (a) What does it mean for a sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X to converge weakly to $p \in X$?

(b) Give an example of a weakly convergent sequence in the sequence space l_2 that is not norm convergent. Fully justify your answer.

Spring 2009 #14. Relative to $[0, 2\pi]$, find the L_1, L_2, and L_∞ norms of $f(x) = \sin x$.